THE CONIC AS A SPACE ELEMENT*

BY

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INTRODUCTION

In this article it is proposed to determine a system of coördinates for the conic in space of three dimensions, analogous to the line coördinates of Plücker, and the similar systems for the circle in two and in three dimensions; and to study systems of conics by means of these coördinates.

The conic in space has been considerably studied in the last sixty years, but no systematic theory has been developed. In 1908, the Belgian Royal Academy announced the offer of a prize for a discussion of the subject, but so far as the writer knows, no award was made.

It will be of interest to summarize briefly the work that has been done in this field.

In 1861 Grunert† treated the general theory of the conic as a space curve, in connection with the study of planetary orbits. Defining a conic by choosing arbitrarily a line as directrix, a point as focus, and the eccentricity, he deduces a number of analytic relations. He also treats a number of other problems, chiefly of interest to the astronomer.

About the same time Chasles‡ laid the foundations of a theory of *characteristics* of conics in space, analogous to the well-known theory for conics in the plane. Hierholzer§ and Lüroth‖ discuss similar problems.

The earliest attempt to determine a set of coördinates for a conic is doubtless that of Spottiswoode.¶ His coördinates are not the simplest nor the most natural ones, however. We shall touch on them later (§ 3, 6, footnote).

A treatment somewhat analogous to parts of ours is given by P. van Geer.** His methods, however, are clumsy, and such of his results as are of value we can obtain by more direct methods.

* Presented to the Society, September 8, 1914.

† Grunert's Archiv, 1861, pp. 1–104.

‡ Comptes Rendus, 1865, pp. 389–397.

§ Mathematische Annalen, vol. 2 (1870), pp. 563–580.

‖ Mathematische Annalen, vol. 3 (1871), pp. 124–133.

¶ Proceedings of the London Mathematical Society, 1878, pp. 185–196.

** Archives Néerlandaises, 1888, pp. 58–90.

Some parts of this paper are based on an article by Reye* on linear systems of quadrics. It will turn out most desirable for us to regard a conic as a special quadric envelope, and hence the methods of studying systems of quadrics are applicable to the study of conics. In the article in question, Reye does not give much special attention to systems of conics; but from his results we can derive at once considerable information concerning linear systems of conics.

For the most part, however, systems of conics have been treated by pure geometrical methods. Several writers have studied particular systems, each defined in a special way. For example, Humbert† discusses the following system: through five fixed points there are ∞^2 cubic curves; of the ∞^1 that touch a fixed plane, the points of contact in general lie on a conic. Thus, a conic is determined in every plane of space; and the paper is a very interesting study of this system. The methods, however, are applicable only to this particular system, and the results no more general than this system. The same remarks apply to the papers of Pieri,‡ de Vries,§ and the first article of Montesano.‖

Since 1890, however, there has been developed a general theory of two and three parameter systems of certain types, the authors being Prof. Domenico Montesano of Pisa and M. Lucien Godeaux of Liège. The former¶ has given a complete treatment of linear congruences,—that is, of two-parameter families with one member through an arbitrary point. In a paper read before the Rome Congress,** he sketched the theory of "bilinear" complexes or three parameter families.

Godeaux, in a number of short notes,†† has considered the same problems. In a later note,‡‡ he called attention to some alleged errors in Montesano's work, and offered corrections. This elicited from Prof. Montesano a rather spirited reply §§ showing deficiencies in the reasoning of M. Godeaux. Apparently they are in agreement as to the facts of the case.

We shall not consider these matters further, for in the present paper we shall adopt a different point of view. Namely, these writers regard a conic as a curve of order 2, while we shall mean by a conic a degenerate envelope of class 2. Hence our totality of conics is a different entity from theirs, and in

*Crelle, Journal für Mathematik, vol. 82 (1876), pp. 54–83.
†Journal de l'école polytechnique, vol. 62 (1894), pp. 123–149.
‡Torino Atti, vol. 28, pp. 289–303.
§Amsterdam Akademie, Verslagen, 1904, pp. 281–284.
‖Rendiconti del Reale Istituto Lombardo, 1893, pp. 589–604.
¶Napoli Rendiconti, 1895–6, pp. 155–181.
**Atti del IV Congresso Internazionale, Roma, 1908, II, pp. 231–233.
††Bulletin de l'Académie de Belgique, 1908–1910.
‡‡Paris Comptes Rendus, vol. 152 (1911), pp. 1149–1151.
§§Paris Comptes Rendus, vol. 153 (1911), pp. 45–48.

dealing with systems of conics, which in general include degenerate members, the results will be different.

In addition to his work on congruences and complexes Godeaux* has attempted to devise coördinates for the conic in space. His most successful method is that of defining a conic as the intersection of a plane with a quadric linearly dependent on six fixed quadrics. The coördinates of the conic are then the four coördinates of the plane and the six multipliers of the system of quadrics. He makes some use of these coördinates in studying linear systems, but to us they seem to have little value. He suggests several other schemes, but they appear to be of even less use than the one we have discussed.

1. Determination of Coördinates

In studying conics in space, it is desirable to determine a system of coördinates, such that to a conic there always corresponds a unique set of coördinates, and conversely.

Two simple methods of defining a conic suggest themselves.

(*a*) If (u_1, u_2, u_3, u_4) be homogeneous coördinates of a plane, the equation

$$\sum_{i,j=1}^{4} a_{ij} u_i u_j = 0 \qquad (a_{ij} = a_{ji}) \tag{1}$$

represents an envelope of class two, which, when the matrix

$$\begin{Vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{Vmatrix} \tag{2}$$

is of rank *three*, is a *conic*. Conversely, any non-degenerate conic of space may be represented by an equation of the type (1) with vanishing discriminant.

This suggests the possibility of taking as coördinates of a conic a set of ten homogeneous numbers

$$(a_{11}, a_{12}, \cdots, a_{44}) \qquad (a_{ij} = a_{ji}), \tag{3}$$

which satisfy the quartic identity

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0. \tag{4}$$

*Bulletin de l'Académie Belgique, 1908, pp. 896–902.

For we have established a one-to-one correspondence between non-degenerate conics and those sets (a) whose matrix is of rank three.

(b) The second method of representing a conic is the parametric form

$$x_i = a_i s^2 + 2b_i st + c_i t^2 \qquad (i = 1, \cdots, 4), \tag{5}$$

where the a's, b's, and c's are constants, and (s, t) are homogeneous parameters. Any set of twelve numbers (a, b, c) define a conic uniquely, in general; but the converse is not true, since these coefficients may be changed by a linear transformation of the parameters which does not change the conic.

The last remark suggests that we may get a desirable set of coördinates by choosing functions of these coefficients that are invariant under all proper linear transformations of the parameters. The simplest invariants of the forms

$$a_i s^2 + 2b_i st + c_i t^2 \qquad (i = 1, \cdots, 4), \tag{6}$$

are the expressions

$$w_{ij} = \tfrac{1}{2}(a_i c_j + c_i a_j - 2b_i b_j) \tag{7}$$

whereof there are sixteen, reducing to ten by virtue of the obvious relation $w_{ij} = w_{ji}$. We have then, for any conic given by (5), a homogeneous set of numbers (7). Clearly they cannot be independent coördinates of a conic; for since the conics of space form an eight-parameter family, we have one superfluous coördinate.

Let us see how the numbers (w) are related to the coördinates indicated under (a). We will find the condition that the conic (5) touches an arbitrary plane

$$(vx) \equiv v_1 x_1 + v_2 x_2 + v_3 x_3 + v_4 x_4 = 0.$$

The two points in which the conic cuts the plane are given by the equation

$$(va)s^2 + 2(vb)st + (vc)t^2 = 0. \tag{8}$$

These points coincide, and the plane touches the conic, if and only if

$$\begin{vmatrix} (va) & (vb) \\ (vb) & (vc) \end{vmatrix} = 0. \tag{9}$$

Expanding this equation, we have by means of (7)

$$\sum_{i,j=1}^{4} w_{ij} v_i v_j = 0, \tag{10}$$

and we see that the ten numbers w_{ij} are precisely the same coördinates to which the first treatment led us. Thus it appears that they are very desirable coördinates for a conic in space.

We next wish to extend our notion of conic in such a way that the correspondence between conics and sets of coördinates shall be without exception. Accordingly, we formulate the following

DEFINITION. *Let*

$$\left\| \begin{matrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{matrix} \right\| \qquad (w_{ji} = w_{ij}) \tag{11}$$

be a set of numbers not all zero, satisfying the relation

$$\begin{vmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{vmatrix} = 0. \tag{12}$$

Then the envelope of class 2, defined by the equation $\Sigma\, w_{ij}\, u_i\, u_j = 0$ *shall be called a* CONIC.

Obviously conics are to be classified according to the rank of the matrix (w).

Case 1. (w) is of rank 3. Then the envelope is a non-degenerate plane conic, or, as we shall say, a *proper conic.*

Case 2. (w) is of rank 2. In this case the left-hand side of equation (12) is reducible into two linear factors, and the planes satisfying (12) are the totality of planes passing through either of two fixed points. Hence the envelope consists of these two points, and we shall call such a conic a *point-pair.*

Case 3. (w) is of rank 1. Then the left-hand side of (12) is the square of a linear form, and the envelope is a single point counted twice. Such a conic we shall call a *point-double.*

From this definition it follows that there is a complete one-to-one correspondence between conics of space and sets of coördinates w with vanishing determinant. But sets of $n + 1$ homogeneous coördinates can always be put in one-to-one correspondence with points of a projective linear space of n dimensions. Hence we have the following representation of conics:

Since a quadric envelope is determined by ten homogeneous coördinates w_{ij}, *the totality of such quadrics correspond in a one-to-one manner with points of a linear space of nine dimensions* S_9. *In this domain the conics of space are represented by points of an eight-dimensional hyperspace of order four, whose equation is given by* (12). *We shall denote this spread* S_8^4 *by* Σ.

2. ELEMENTARY PROPERTIES

1. Consider a conic (w) of rank 3. Write

$$\Delta(w) \equiv \begin{vmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ w_{21} & w_{22} & w_{23} & w_{24} \\ w_{31} & w_{32} & w_{33} & w_{34} \\ w_{41} & w_{42} & w_{43} & w_{44} \end{vmatrix} = 0. \tag{1}$$

We shall denote the co-factor of any element w_{ij} of this determinant by W_{ij}. Then by hypothesis the W's do not all vanish, and

$$\Delta \equiv \sum_{i=1}^{4} w_{ij} W_{ij} \equiv \sum_{j=1}^{4} w_{ij} W_{ij}. \tag{2}$$

THEOREM. *The adjoint matrix*

$$\begin{Vmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{Vmatrix} \tag{3}$$

is of rank unity.

For at least one member is not zero, by hypothesis; but every two-rowed determinant of (3) has Δ as a factor, and hence vanishes.

It follows that the W's are dependent on four parameters; and in fact, if we set

$$W_{11} = D_1^2, \qquad W_{12} = D_1 D_2, \qquad W_{13} = D_1 D_3, \qquad W_{14} = D_1 D_4 \tag{4}$$

we obtain

$$W_{ij} = D_i D_j \qquad (i, j = 1, \cdots, 4). \tag{5}$$

We can show at once that the D's so defined are the coördinates of the plane of the conic. For in general the quadric

$$\sum_{i,j=1}^{4} w_{ij} u_i u_j = 0 \tag{6}$$

has one and only one singular plane, if the system

$$\begin{aligned} w_{11} u_1 + w_{12} u_2 + w_{13} u_3 + w_{14} u_4 &= 0, \\ w_{21} u_1 + w_{22} u_2 + w_{23} u_3 + w_{24} u_4 &= 0, \\ w_{31} u_1 + w_{32} u_2 + w_{33} u_3 + w_{34} u_4 &= 0, \\ w_{41} u_1 + w_{42} u_2 + w_{43} u_3 + w_{44} u_4 &= 0, \end{aligned} \tag{7}$$

has one and essentially only one solution not consisting of zeros; i. e., if the matrix (w) is of rank 3. When this occurs, there are three of the equations (7) that are linearly independent; and by solving them we get the coördinates of the plane. For example, if $W_{11} \neq 0$, so that the last three equations are independent, we have from these three

$$u_1 : u_2 : u_3 : u_4 = W_{11} : W_{12} : W_{13} : W_{14} = D_1 : D_2 : D_3 : D_4, \tag{8}$$

as was to be shown.

2. Consider next a conic (w) of rank 2. In this case, the equations (7)

are equivalent to *two* equations; and therefore the system has ∞^1 solutions, linearly dependent on two. That is to say, a conic of rank two has a pencil of singular planes.

We have noted that in this case the envelope consists of two points; and it is obvious that the singular planes are the planes through both of these points. Hence the axis of the pencil is the line joining the two points. We shall call this line the *axis of the conic.* Its coördinates may be found as follows.

Each of the equations (7), in which not all the coefficients vanish, is the equation of a point, whose coördinates are the coefficients of the u's. Further, since every singular plane satisfies (7), these points lie on the axis of the conic. For definiteness assume that

$$\begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix} \neq 0.$$

Then $(w_{11}, w_{12}, w_{13}, w_{14})$ and $(w_{21}, w_{22}, w_{23}, w_{24})$ are coördinates of two distinct points on the axis, and the plücker coördinates of the latter are accordingly given by

$$\rho_{12}\, p_{ij} = \begin{vmatrix} w_{1i} & w_{1j} \\ w_{2i} & w_{2j} \end{vmatrix} \qquad (i = 1, \cdots, 4), \tag{9}$$

where ρ_{12} is a constant factor not zero.

Similarly, for any two distinct equations of (7),

$$\rho_{mn}\, p_{ij} = \begin{vmatrix} w_{mi} & w_{mj} \\ w_{ni} & w_{nj} \end{vmatrix} \qquad (i, j = 1, \cdots, 4). \tag{10}$$

To determine ρ_{mn}, set $i = 1, j = 2$. Then

$$\rho_{mn}\, p_{12} = \begin{vmatrix} w_{m1} & w_{m2} \\ w_{n1} & w_{n2} \end{vmatrix} \equiv \begin{vmatrix} w_{1m} & w_{1n} \\ w_{2m} & w_{2n} \end{vmatrix} = \rho_{12}\, p_{mn}, \tag{11}$$

whence if we take $\rho_{12} = p_{12}$, we have $\rho_{mn} = p_{mn}$.

Obviously also when $p_{mn} = 0$, then $\rho_{mn} = 0$ also; and we have the

THEOREM. *Let (w) be a conic of rank two, and let (p) be the plücker coördinates of its axis. Then*

$$p_{ij}\, p_{mn} = \begin{vmatrix} w_{im} & w_{in} \\ w_{jm} & w_{jn} \end{vmatrix}. \tag{12}$$

3. Consider a conic (w) of rank 1. We have

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j \equiv (x_1\, u_1 + x_2\, u_2 + x_3\, u_3 + x_4\, u_4)^2 \tag{13}$$

and hence

$$w_{ij} = x_i\, x_j. \tag{14}$$

Hence when a conic is a point-double, we can at once write the coördinates of its point; and conversely, we have at once the w-coördinates of the point-double at any point.

3. Relations to Point, Line, and Plane

It is now proposed to determine the representation in S_9 of simple systems of conics in space.

1. First let us recall that a set of coördinates (w) define on the one hand a quadric envelope, and on the other hand a point of a nine dimensional space S_9.

If, in particular,

$$\Delta(w) = 0, \tag{1}$$

the envelope is a conic, and the corresponding point lies on a quartic hypersurface which we shall call Σ.

2. The condition that a conic be tangent to a plane (u) is obviously

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0. \tag{2}$$

When (u) is given, this is linear in (w). Hence we have the result:

The totality of conics that touch a fixed plane (u) are represented in S_9 by the intersection of Σ with a hyperplane whose equation is (2).

Evidently there are ∞^3 such hyperplanes. We will now show that they are precisely the tangent hyperplanes to Σ. For the equation of Σ is $\Delta(w) = 0$, and the polar hyperplane, with regard to Σ, of any point $(\bar{w})$ is

$$\sum_{i,j=1}^{4} \frac{\partial \Delta(\bar{w})}{\partial \bar{w}_{ij}}\, w_{ij} = 0;$$

and this may be written

$$\sum_{i,j=1}^{4} \overline{W}_{ij}\, w_{ij} = 0. \tag{3}$$

In particular, this hyperplane contains $(\overline{w})$ itself if and only if $\Delta(\bar{w}) = 0$, and hence $(\bar{w})$ is a point on Σ. In this case $(\bar{w})$ corresponds to a conic, and the plane of the latter is given by $\overline{W}_{ij} = v_i\, v_j$. Hence at any point $(\bar{w})$ of Σ, the tangent hyperplane is given by

$$\sum_{i,j=1}^{4} v_i\, v_j\, w_{ij} = 0, \tag{4}$$

where (v) is the plane of the conic $(\bar{w})$. It follows that *the hypersurface Σ has the same tangent hyperplane at all points which correspond to conics in any given plane; and hence it has only ∞^3 tangent hyperplanes, each touching it at ∞^5 points.* The S_7^4 (spread of 7 dimensions and order 4) in which a tangent hyperplane cuts Σ corresponds to the totality of conics that touch the plane.

It is clear that these remarks apply only to points of Σ that correspond to proper conics; for at other points the tangent hyperplane is indeterminate. That is, the points that correspond to point-pairs and point-doubles generate a singular locus on Σ; but at ordinary points Σ is of a nature analogous to that of a developable surface. To complete the analogy, we observe that the ∞^5 points of tangency of a tangent hyperplane form a *linear* spread. For the conditions that the plane (v) be a singular plane of the envelope

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0$$

are clearly the four equations

$$w_{i1}\, v_1 + w_{i2}\, v_2 + w_{i3}\, v_3 + w_{i4}\, v_4 = 0 \qquad (i = 1, \cdots, 4) \tag{5}$$

If (v) be given, and the w's be regarded as independent variables except for the conditions $w_{ij} = w_{ji}$, then these equations are independent. From them, however, $\Delta(w) = 0$ is a consequence, so that any quadric (w) which satisfies (5) is a conic in the plane (v). Hence the conics in a given plane are the totality of solutions of four linear equations, and so they compose a linear five-parameter family S_5. Summarizing, we have the

THEOREM. *The points of S_9 corresponding to the conics in a given plane form a linear S_5. Through each ordinary point of Σ there passes one and only one such S_5, which is the totality of points of contact of the tangent hyperplane to Σ at that point.*

3. In an article on linear systems of quadrics,* Reye deduces a number of results which give us at once several important theorems. He determines the dimension (Stufe) and order (Grad) of the family of conics which satisfy a given condition; from which we have at once the nature of the spread in S_9 that represents the family. We shall not reproduce his work, which is rather long; but presently we shall summarize his results.

4. We next determine the equation, in point coördinates, of the cone from a point (y) to a conic (w).

First let us find the cone from a fixed point (y) that envelopes a proper quadric (a)

$$\sum_{i,j=1}^{4} a_{ij}\, u_i\, u_j = 0. \tag{6}$$

Let (x) be any point, then the condition that a point $\lambda(x) + \mu(y)$ on the line (yx) also lie on the quadric (6) is found by means of the point-equation of the latter,

* Loc. cit., Introduction.

$$(7)\qquad \begin{vmatrix} 0 & \lambda x_1+\mu y_1 & \lambda x_2+\mu y_2 & \lambda x_3+\mu y_3 & \lambda x_4+\mu y_4 \\ \lambda x_1+\mu y_1 & a_{11} & a_{12} & a_{13} & a_{14} \\ \lambda x_2+\mu y_2 & a_{21} & a_{22} & a_{23} & a_{24} \\ \lambda x_3+\mu y_3 & a_{31} & a_{32} & a_{33} & a_{34} \\ \lambda x_4+\mu y_4 & a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0.$$

This is quadratic in $\lambda : \mu$. Its roots will be equal and the line will touch the quadric, if and only if

$$(8)\qquad \begin{vmatrix} 0 & y_1 & y_2 & y_3 & y_4 \\ x_1 & a_{11} & \cdot & \cdot & a_{14} \\ x_2 & \cdot & \cdot & \cdot & \cdot \\ x_3 & \cdot & \cdot & \cdot & \cdot \\ x_4 & a_{41} & \cdot & \cdot & a_{44} \end{vmatrix}^2 - \begin{vmatrix} 0 & x_1 & x_2 & x_3 & x_4 \\ x_1 & a_{11} & \cdot & \cdot & a_{14} \\ x_2 & \cdot & \cdot & \cdot & \cdot \\ x_3 & \cdot & \cdot & \cdot & \cdot \\ x_4 & a_{41} & \cdot & \cdot & a_{44} \end{vmatrix} \begin{vmatrix} 0 & y_1 & y_2 & y_3 & y_4 \\ y_1 & a_{11} & \cdot & \cdot & a_{14} \\ y_2 & \cdot & \cdot & \cdot & \cdot \\ y_3 & \cdot & \cdot & \cdot & \cdot \\ y_4 & a_{41} & \cdot & \cdot & a_{44} \end{vmatrix} = 0.$$

Since $\Delta(a) \neq 0$, this equation is equivalent to the following (cf. Bôcher's *Higher Algebra*, p. 31).

$$(9)\qquad \begin{vmatrix} 0 & 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ x_1 & y_1 & a_{11} & a_{12} & a_{13} & a_{14} \\ x_2 & y_2 & a_{21} & a_{22} & a_{23} & a_{24} \\ x_3 & y_3 & a_{31} & a_{32} & a_{33} & a_{34} \\ x_4 & y_4 & a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0,$$

and this is accordingly the equation of the cone.

If we start with a proper conic instead of a quadric, this method fails. Let the conic be

$$(10)\qquad \sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0, \qquad \Delta(w) = 0.$$

We form the equation

$$(11)\qquad \sum_{i,j=1}^{4} (w_{ij} + \rho y_i\, y_j)\, u_i\, u_j = 0, \qquad \rho \neq 0,$$

which is a quadric through (w) and having the point (y) as vertex of the tangent cone along (w). Setting $a_{ij} = w_{ij} + \rho y_i\, y_j$, we have

$$(12)\qquad \Delta(a) \equiv \Delta(w) + \rho \sum_{i,j=1}^{4} y_i\, y_j\, W_{ij} \equiv \rho \sum_{i,j=1}^{4} d_i\, d_j\, y_i\, y_j \equiv \rho\,(dy)^2.$$

If (y) is not in the plane (d) of (w), and we take $\rho \neq 0$, then $\Delta(a) \neq 0$, and (a) is a proper quadric. Applying the previous result, we get the desired equation in the form

$$(13)\qquad \begin{vmatrix} 0 & 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ x_1 & y_1 & w_{11}+\rho y_1^2 & w_{12}+\rho y_1 y_2 & w_{13}+\rho y_1 y_3 & w_{14}+\rho y_1 y_4 \\ x_2 & y_2 & w_{21}+\rho y_2 y_1 & \cdot & \cdot & \cdot \\ x_3 & y_3 & w_{31}+\rho y_3 y_1 & \cdot & \cdot & \cdot \\ x_4 & y_4 & w_{41}+\rho y_4 y_1 & \cdot & \cdot & \cdot \end{vmatrix} = 0.$$

This may be simplified by subtracting multiples of the second row from later rows in an obvious manner. This leads to the

THEOREM. *The equation of the cone drawn to any proper conic from a point* (y) *not in its plane is*

$$(14)\qquad \begin{vmatrix} 0 & 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & 0 & y_1 & y_2 & y_3 & y_4 \\ x_1 & y_1 & w_{11} & \cdot & \cdot & w_{14} \\ x_2 & y_2 & \cdot & \cdot & \cdot & \cdot \\ x_3 & y_3 & \cdot & \cdot & \cdot & \cdot \\ x_4 & y_4 & w_{41} & \cdot & \cdot & w_{44} \end{vmatrix} = 0.$$

Remark. It is easily shown that if (y) be a point in the plane of (w), then (14) becomes $(yx)^2 = 0$, where $(yx) = 0$ is the equation of the plane.

5. We shall next determine the order of the S_6 that represents conics through a fixed point.

In the first place, if (w) be a conic whose plane does not pass through $(0, 0, 0, 1)$, then the cone from this point to (w) is given by

$$(15)\qquad \begin{vmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & w_{11} & w_{12} & w_{13} \\ x_2 & w_{21} & w_{22} & w_{23} \\ x_3 & w_{31} & w_{32} & w_{33} \end{vmatrix} = 0.$$

Conversely, if $P:(x_1, x_2, x_3, x_4)$ be a fixed point, the conics (w) that satisfy (15) are those that meet the line from $P_0:(0, 0, 0, 1)$ to P. For

(a) if $W_{44} \neq 0$, the plane of the conic does not pass through P_0: and since P lies on the cone, the element $P_0 P$ of the cone meets the conic;

(*b*) if $W_{44} = 0$, (15) is the equation of the plane of (w); and since both P and P_0 lie in this plane, their line meets (w) twice.

Now a conic passes through P if and only if

(1) its plane passes through P, and (2) it intersects one line through P, not lying in its plane.

We shall consider the line $P_0 P$. The conics that meet it satisfy (15); and this defines in S_9 a hypersurface S_8^2. The conics whose planes pass through a fixed point P form, as Reye (loc. cit.) shows, a S_7^6. The points common to a S_8^2 and a S_7^6 form a S_6^{12}. We have just seen that these include two sets of conics: first, the conics through P, and second, those whose planes pass through $P_0 P$. The latter, as Reye (loc. cit.) shows, correspond to a S_6^4. Hence the conics through P are represented by a S_6^8, and we have the

THEOREM. *The conics which pass through a given point P are represented in S_9 by a S_6^8.*

6. By methods similar to those of part 4, we can prove the

THEOREM. *Let*

$$\sum_{i,j=1}^{4} a_{ij}\, u_i\, u_j = 0$$

be any quadric envelope. Let the plücker coördinates of a line be given by

$$p_{01} = \pi_{23}, \qquad p_{02} = \pi_{31}, \qquad p_{03} = \pi_{12}, \qquad \text{etc.}$$

Then the equation

$$\sum_{1}^{4} \begin{vmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{vmatrix} \pi_{ik}\, \pi_{jl} = 0$$

has the following interpretations:

(*a*) *If (a) is a proper quadric, (π) touches it.*

(*b*) *If (a) is a proper conic, (π) meets it.*

(*c*) *If (a) is a point-pair, (π) meets its axis.*

(*d*) *If (a) is a point-double, the equation is satisfied identically.*

For convenience we make the following

DEFINITION. *A line (π) shall be said to intersect a conic that is a point-pair, when it intersects the axis of the latter.*

We have then the

THEOREM. *The totality of conics that intersect a given line (π) are represented in S_9 by the total intersection of Σ with the S_8^2 whose equation is*

$$\sum_{1}^{4} \begin{vmatrix} w_{ij} & w_{il} \\ w_{kj} & w_{kl} \end{vmatrix} \pi_{ik}\, \pi_{jl} = 0.^{*} \tag{16}$$

* In equation (16), if (w) be regarded as fixed, the equation is that of the complex of lines that meet the conic (w). The coefficients may thus be regarded as coördinates of the conic. This is essentially the system of coördinates used by Spottiswoode. (See Introduction.)

There are ∞^4 such S_8^2's, corresponding to the lines of space; and they have in common the singular locus (see below) that represents point-doubles.

7. We will close the section with a tabular summary of the correspondence we have developed, including the results obtained by Reye. The latter are indicated by the symbol #.

In S_3	In S_9
Quadric envelope	Point
Conic	Point of $S_9^1 \equiv \Sigma$
Conics in fixed plane	S_8 lying in Σ
Quadrics touching a fixed plane	S_8. Touching Σ at all points of a S_8
Conics whose planes pass through fixed point	S_7^6 #
Conics whose planes pass through fixed line	S_6^4 #
Conics which pass through fixed point	S_6^3 (5)
Conics that meet a fixed line	S_7^3 (6)
Point-pairs	S_6^{10} #
Point-pairs whose lines meet a fixed line	S_5^{10} #
Point-pairs with one member on given line	S_4^4 #
Point-pairs with both members on given line	S_2^1 #
Point-pairs with both members in given plane	S_4^3 #
Point-doubles	S_3^3 #
Point-doubles in given plane	S_2^4 #
Point-doubles on given line	S_1^2 #

4. Relations Between Conic and Quadric

Let us determine the conic in which a given plane cuts a given quadric. We observe that this is exactly dual to a problem already treated (§ 3, 4); hence we need not repeat the work. The result is as follows:

Theorem. *The condition that a plane* (u) *touch the conic* (w) *in which a given quadric*

$$\sum_{i,j=1}^{4} a_{ij}\, x_i\, x_j = 0 \tag{1}$$

is cut by a given plane (A_1, A_2, A_3, A_4) *is*

$$\begin{vmatrix} 0 & 0 & u_1 & u_2 & u_3 & u_4 \\ 0 & 0 & A_1 & A_2 & A_3 & A_4 \\ u_1 & A_1 & a_{11} & a_{12} & a_{13} & a_{14} \\ u_2 & A_2 & a_{21} & a_{22} & a_{23} & a_{24} \\ u_3 & A_3 & a_{31} & a_{32} & a_{33} & a_{34} \\ u_4 & A_4 & a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = 0. \tag{2}$$

Hence the coördinates of the conic are immediately written down. If, in particular, the plane (A) touches the quadric, equation (2) reduces to the equation of the point of contact, counted twice.

We next determine conditions that a conic lie on a given quadric. These conditions do not seem capable of very simple expression. We will treat the problem in two ways. Of these, the former gives a redundant set of conditions; but as usual in such cases, it is impossible to choose a set which will in all cases be sufficient and independent. The second method, on the other hand, gives exactly the right number of equations; but they are complicated in form.

FIRST METHOD. If

$$\sum_{i,j=1}^{4} A_{ij}\, u_i\, u_j = 0 \tag{3}$$

be a proper quadric, and

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0$$

a conic lying on it, then one member of the pencil linearly dependent on them will be the pole of the plane of (w), counted twice. (This is well known in the theory of quadrics.) Hence (w) lies on (A) if and only if there is a number k such that the matrix

$$\left\| \begin{matrix} A_{11}+kw_{11} & A_{12}+kw_{12} & A_{13}+kw_{13} & A_{14}+kw_{14} \\ A_{21}+kw_{21} & \cdot & \cdot & \cdot \\ A_{31}+kw_{31} & \cdot & \cdot & \cdot \\ A_{41}+kw_{41} & \cdot & \cdot & \cdot \end{matrix} \right\| \tag{4}$$

is of rank 1. This gives a number of quadratic equations in k, of the type

$$k^2 \begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix} + k \left\{ \begin{vmatrix} w_{11} & A_{12} \\ w_{21} & A_{22} \end{vmatrix} + \begin{vmatrix} A_{11} & w_{12} \\ A_{12} & w_{22} \end{vmatrix} \right\} + \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = 0. \tag{5}$$

The required conditions are that these equations all have a common root, and we may obtain them by equating to zero the resultant of each pair of them. These equations are necessary and sufficient, but are not independent.

SECOND METHOD. This consists in making a non-singular linear transformation that carries the plane of the conic into the plane $(0,0,0,1)$. Let a proper quadric be given by the point equation

$$\sum_{i,j=1}^{4} a_{ij}\, x_i\, x_j = 0, \qquad \Delta(a) \neq 0, \tag{6}$$

and hence by the tangential equation

$$\sum_{i,j=1}^{4} A_{ij}\, u_i\, u_j = 0, \tag{7}$$

where, as usual, A_{ij} is the cofactor of a_{ij} in $\Delta(a)$. Let (w) be a conic,

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0, \qquad \Delta(w) = 0, \tag{8}$$

and denote the coördinates of its plane by (d). Then

$$W_{ij} = d_i\, d_j, \qquad \sum_{i=1}^{4} w_{ij}\, d_i = 0 \qquad (j = 1, \cdots, 4). \tag{9}$$

Case 1. $d_4 = 0$. Then without making the transformation indicated under case 2, we can carry on the later work directly, as will be obvious.

Case 2. $d_4 \neq 0$. The linear transformation

$$\left.\begin{aligned} u_1' &= d_4\, u_1 && - d_1\, u_4 \\ u_2' &= d_4\, u_2 && - d_2\, u_4 \\ u_3' &= d_4\, u_3 && - d_3\, u_4 \\ u_4' &= && - d_4\, u_4 \end{aligned}\right\} \tag{10}$$

is non-singular and involutory. We will apply it to the quadratic equations

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0 \tag{11}$$

$$\sum_{i,j=1}^{4} A_{ij}\, u_i\, u_j = 0. \tag{12}$$

We find that the former goes into

$$\sum_{i,j=1}^{3} w_{ij}\, u_i'\, u_j' = 0, \tag{13}$$

while the latter goes into an equation whose matrix is

$$\left\| \begin{array}{cccc} d_4^2 A_{11} & d_4^2 A_{12} & d_4^2 A_{13} & -d_4 \sum_{i=1}^{4} d_i A_{i1} \\ d_4^2 A_{21} & d_4^2 A_{22} & d_4^2 A_{23} & -d_4 \sum_{i=1}^{4} d_i A_{i2} \\ d_4^2 A_{31} & d_4^2 A_{32} & d_4^2 A_{33} & -d_4 \sum_{i=1}^{4} d_i A_{i3} \\ -d_4 \sum_{i=1}^{4} d_i A_{i1} & -d_4 \sum_{i=1}^{4} d_i A_{i2} & -d_4 \sum_{i=1}^{4} d_i A_{i3} & \sum_{i,j=1}^{4} d_i\, d_j\, A_{ij} \end{array} \right\|, \tag{14}$$

so that this equation may be written

$$d_4^2 \sum_{i,j=1}^{3} A_{ij}\, u_i'\, u_j' - 2 d_4 \sum_{\substack{i=1,\, j=3 \\ i=1 \\ j=1}}^{} d_i\, A_{ij}\, u_j'\, u_4' + \left(\sum_{i,j=1}^{4} d_i\, d_j\, A_{ij} \right) u_4'^2 = 0. \tag{15}$$

We shall for brevity represent this equation by

$$\sum_{i,j=1}^{4} k_{ij}\, u_i' u_j' = 0,$$

and denoting the cofactors of the elements k_{ij} of (14) by K_{ij}, we get the result:

The effect of the transformation (10) on the quadric

$$\sum_{i,j=1}^{4} a_{ij}\, x_i x_j = 0 \tag{7}$$

is to carry it into the quadric

$$\sum_{i,j=1}^{4} K_{ij}\, x_i' x_j' = 0. \tag{16}$$

We next find the conic in which this quadric cuts the plane $(0, 0, 0, 1)$. In formula (2) set $(A) \equiv (0, 0, 0, 1)$, and $a_{ij} \equiv K_{ij}$. We get

$$\begin{vmatrix} 0 & u_1' & u_2' & u_3' \\ u_1' & K_{11} & K_{12} & K_{13} \\ u_2' & K_{21} & K_{22} & K_{23} \\ u_3' & K_{31} & K_{32} & K_{33} \end{vmatrix} = 0 \tag{17}$$

as the equation of the conic. But

$$\begin{vmatrix} K_{ij} & K_{im} \\ K_{nj} & K_{nm} \end{vmatrix} = M\Delta(k), \tag{18}$$

where M is the algebraic complement of $k_{ij}\, k_{mn}$ in $\Delta(k)$. Hence (17) reduces to

$$\sum_{i,j=1}^{3} \begin{vmatrix} k_{ij} & k_{i4} \\ k_{j4} & k_{44} \end{vmatrix} u_i' u_j' = 0, \tag{19}$$

and this is the conic in which the quadric cuts the plane of the given conic, after the transformation (10).

This coincides with the given conic if and only if

$$\rho w_{ij} = \begin{vmatrix} k_{ij} & k_{i4} \\ k_{j4} & k_{44} \end{vmatrix} \qquad (i, j = 1, 2, 3), \tag{20}$$

or, when we replace the k's by their values from (14), and set $d_i\, d_j = W_{ij}$,

$$\rho w_{ij} = \begin{vmatrix} W_{44}\, A_{ij} & -\sum_{l=1}^{4} W_{l4}\, A_{lj} \\ -\sum_{l=1}^{4} W_{4l}\, A_{li} & \sum_{l,m=1}^{4} W_{lm}\, A_{lm} \end{vmatrix} \qquad (i, j = 1, 2, 3). \tag{21}$$

These equations are six in number, but they involve an inessential factor ρ. By eliminating it, we get exactly the correct number of equations, namely five.

5. Relations Between Two Conics

Let (a) and (b) be any two conics. The quadrics linearly dependent on them,

$$\sum_{j=1}^{4} (\lambda a_{ij} + \mu b_{ij}) u_i u_j = 0 \tag{1}$$

form a one-parameter family called a *series*. Among them there are in general four conics, given by values of $\lambda : \mu$ that satisfy the equation

$$\Delta(\lambda a + \mu b) = 0. \tag{2}$$

It is known that this equation is invariant under all proper collineations; hence the vanishing of any one of its coefficients is the condition for some special projective relation between the two given conics.

We shall represent (2) by the equation

$$\lambda^4 \Delta(a) + \lambda^3 \mu \Theta(a, b) + \lambda^2 \mu^2 \Phi(a, b) + \lambda \mu^3 \Theta(b, a) + \mu^4 \Delta(b) = 0 \tag{3}$$

and shall refer to the functions $\Theta(a, b)$, $\Theta(b, a)$, $\Phi(a, b)$ as the *invariants* of the two conics. We observe that the coefficients of λ^4 and μ^4 are the discriminants of the conics (a) and (b) respectively, and hence are vanishing invariants. We propose to investigate the meaning of the vanishing of the invariants Θ and Φ.

These expressions are

$$\Theta(a, b) = \sum_{i,j=1}^{4} b_{ij} A_{ij}, \qquad \Theta(b, a) = \sum_{i,j=1}^{4} a_{ij} B_{ij},$$

$$\Phi(a, b) = \Phi(b, a) = \sum \begin{vmatrix} b_{11} & b_{12} & a_{13} & a_{14} \\ b_{21} & b_{22} & a_{23} & a_{24} \\ b_{31} & b_{32} & a_{33} & a_{34} \\ b_{41} & b_{42} & a_{43} & a_{44} \end{vmatrix}, \tag{4}$$

the summation being effected by replacing in turn every pair of columns of a's by the corresponding b's, and adding all these determinants.

1. Suppose $\Theta(a, b) = 0$.

Case a. Let (a) be a proper conic. Denoting the coördinates of its plane by $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, we have $A_{ij} = \alpha_i \alpha_j$. Hence

$$\Theta(a, b) = \sum_{i,j=1}^{4} b_{ij} A_{ij} = \sum_{i,j=1}^{4} b_{ij} \alpha_i \alpha_j,$$

and so when $\Theta(a, b)$ vanishes, we have

$$\sum_{i,j=1}^{4} b_{ij}\,\alpha_i\,\alpha_j = 0, \tag{5}$$

and (α) is one of the planes that touch the envelope (b).

Case b. Let (a) be a degenerate conic. Then $A_{ij} = 0$ $(i, j = 1, \cdots, 4)$, and for every conic (b) we have $\Theta(a, b) = 0$.

Remark. It is obvious that the equation

$$\sum_{i,j=1}^{4} b_{ij}\,\alpha_i\,\alpha_j = 0$$

is satisfied whenever the conics (a), (b) are coplanar.

We have then the result:

THEOREM. *The equation $\Theta(a, b) = 0$ is satisfied if and only if one of the following occurs*:

(1) *The conics (a) and (b) are coplanar;* (2) *(a) is a proper conic whose plane touches (b);* (3) *(a) is a degenerate conic.*

2. If now we interchange (a) and (b), we have the interpretation of $\Theta(b, a)$.

3. We next consider the equation

$$\Phi(a, b) = 0.$$

Case a. The conics are coplanar. We have already mentioned that the functions Θ, Φ are invariant under any non-singular linear transformation. If we make such a transformation, carrying the common plane of the conics into the plane $(0, 0, 0, 1)$, their matrices assume the forms

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & 0 \\ a'_{21} & a'_{22} & a'_{23} & 0 \\ a'_{31} & a'_{32} & a'_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} b'_{11} & b'_{12} & b'_{13} & 0 \\ b'_{21} & b'_{22} & b'_{23} & 0 \\ b'_{31} & b'_{32} & b'_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and we see by inspection that $\Phi(a', b') = 0$. Hence the invariant Φ of two coplanar conics vanishes.

Case b. If either conic is a point pair, then as we saw in 2, the two-rowed determinants of its matrix are expressible as products of the plücker coördinates of its axis. If (b) is a point-pair, then $\Phi(a, b) = 0$ reduces to

$$\sum_{1}^{4} \begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix} \pi_{ij}\,\pi_{kl} = 0,$$

where the π's are plücker coördinates of the axis of (b). But this (§ 3) is the condition that the line (π) meet the conic (a) when the latter is a

proper conic; and if it is a point-pair, the equation means that the axes of the two point-pairs intersect.

If either conic is a point-double, Φ vanishes identically.

Case c. The remaining case is that neither of the conics is degenerate, and they lie in distinct planes.

To simplify the work, we may without loss of generality take as the planes of the conics the planes $(0,0,0,1)$ and $(1,0,0,0)$ respectively. Then their matrices are

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b_{22} & b_{23} & b_{24} \\ 0 & b_{32} & b_{33} & b_{34} \\ 0 & b_{42} & b_{43} & b_{44} \end{bmatrix},$$

and $\Phi = 0$, reduces to the expression

$$(10) \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \cdot \begin{vmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{vmatrix} - 2 \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{32} \end{vmatrix} \cdot \begin{vmatrix} b_{23} & b_{24} \\ b_{43} & b_{44} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \begin{vmatrix} b_{22} & b_{24} \\ b_{42} & b_{44} \end{vmatrix} = 0.$$

We will determine the pair of points in which the line common to the planes of the conics cuts the conic (a).

The cone from the point $(0,0,0,1)$ to the conic (a) is (§ 3)

$$(11) \qquad \begin{vmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & a_{11} & a_{12} & a_{13} \\ x_2 & a_{21} & a_{22} & a_{23} \\ x_3 & a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

This is cut by the line $x_1 = 0$, $x_4 = 0$ in the two points found by setting $x_1 = 0$ in this equation. We have

$$(12) \qquad x_2^2 \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - 2x_2 x_3 \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + x_3^2 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0.$$

Similarly, the points in which (b) cuts this line are given by the equation

$$(12') \qquad x_2^2 \begin{vmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{vmatrix} - 2x_2 x_3 \begin{vmatrix} b_{23} & b_{24} \\ b_{43} & b_{44} \end{vmatrix} + x_3^2 \begin{vmatrix} b_{22} & b_{24} \\ b_{42} & b_{44} \end{vmatrix} = 0.$$

Then the well-known condition that the roots of two quadratic equations separate each other harmonically gives us

$$\Phi(a, b) = 0$$

by means of equation (10).

Summarizing, we have the

THEOREM. *Two conics* (a) *and* (b) *satisfy the relation* $\Phi(a,b)=0$ *if and only if one of the following occurs*:

(1) *They are coplanar.* (2) *They are proper conics in different planes, and separate each other harmonically on the line of intersection of their planes.* (3) *One of them is a point-pair, and its axis meets the other.* (4) *They are both point-pairs, with intersecting axes (special case of* (1)). (5) *One conic is a point-double.*

4. The condition that two conics intersect at a point is simply expressed in terms of these invariants. Salmon* shows that if two cones have a common tangent plane, the two remaining cones of their pencil fall together.

Dually, if two conics have a common point, the other two conics linearly dependent on them will fall together, and conversely.

Let (a) and (b) be two conics, and let

$$c_{ij}=\lambda a_{ij}+\mu b_{ij}.$$

The four conics of the series (c) are given by

$$\Delta(c)=0,$$

or, since we know that (a) and (b) are two of them, the others are given by

$$\lambda^2\Theta(a,b)+\lambda\mu\,\Phi(a,b)+\mu^2\Theta(b,a)=0.$$

As we have remarked, (a) and (b) intersect if and only if this equation has equal roots, so that we have the result:

THEOREM. *Two proper conics,* (a) *and* (b), *intersect if and only if*

$$\begin{vmatrix} 2\Theta(a,b) & \Phi(a,b) \\ \Phi(a,b) & 2\Theta(b,a) \end{vmatrix}=0.$$

It is seen that this equation is quartic in the coördinates of either conic; hence the conics that meet a fixed conic are represented in S_9 *by a* S_7^{16}, *the total intersection of* Σ *with a* S_8^1.

6. COMPLETE AND INCOMPLETE SYSTEMS—THE INCOMPLETE LINEAR SYSTEMS

We have seen that every set of numbers $(w_{11}, w_{12}, \cdots, w_{44})$, $(w_{ij}=w_{ji})$, are the tangential coördinates of a quadric; and in particular, any set that satisfy the relation

(1) $$\Delta(w)=0$$

* *Geometry of Three Dimensions*, 3d ed., § 202.

are coördinates of a conic. Any set of equations on the w's, of which $9 - r$ are independent, determine an r-parameter system of quadrics, and of these, the ones that satisfy (1) are conics. Hence in studying systems of conics defined by equations, we recognize two distinct cases.

Case 1. The equation (1) is not a necessary consequence of the given equations. Then the latter are satisfied by ∞^r quadrics, which include ∞^{r-1} conics given by the further equation (1). Such a system of conics we shall call an *incomplete system.*

Case 2. The equation (1) is satisfied by every solution of the given equations. Then every one of the solutions of the latter is a conic, and the system consists of ∞^r conics. A system of this sort we shall call a *complete system.*

In the nine-dimensional representation of conics and quadrics, a set of $9 - r$ independent equations define a spread of dimension r. In case this does not lie entirely on Σ, we have an incomplete system, represented by its intersection with Σ. In case we have a complete system, the S_r lies entirely on Σ.

Linear systems constitute an important case. The quadrics that satisfy $9 - r$ linearly independent linear equations may be given as linearly dependent on $r + 1$ quadrics and are represented in S_9 by a linear S_r. In general, this will cut Σ in a S^4_{r-1}, and we have an incomplete linear system; but it may happen that the S_r lies on Σ, so that every solution

$$w_{ij} = \lambda_1 a_{ij}^{(1)} + \cdots + \lambda_{r+1} a_{ij}^{(r+1)} \tag{2}$$

satisfies (1), and then we have a complete linear system.

We have already seen the existence of a complete linear system, namely the ∞^5 conics in a plane. In the following section we shall determine all possible complete linear systems. We take up first incomplete systems.

Evidently it is desirable to study these by means of linear systems of quadrics. The latter have been somewhat studied, and in the simplest cases they are well known. The general one, two and three parameter systems of quadric envelopes are treated in the *Encyklopädie*, III C2, § VIII, § XII, § XIII. The remaining cases are briefly treated in the same place, § XIV. The most important work on the subject is that of Reye, in the paper to which we have already referred (Introduction). In this paper he considers linear systems of point-quadrics and of envelope quadrics of all dimensions from one up to eight. From his results, obviously, we can get at once important facts concerning the systems of conics included in his systems of quadrics, though he says little specifically about these conics.

Before considering linear systems, we need to discuss the relation of *apolarity*, which is closely related.

DEFINITION. *Let*

$$\sum_{i,j=1}^{3} a_{ij}\, x_i\, x_j = 0, \tag{3}$$

$$\sum_{i,j=1}^{3} b_{ij}\, u_i\, u_j = 0 \tag{4}$$

be two conics in a plane. If the relation

$$\sum_{i,j=1}^{3} a_{ij}\, b_{ij} = 0 \tag{5}$$

is satisfied, the conics are said to be APOLAR.*

The geometrical meaning of the relation is that there exists a triangle self-conjugate with regard to (a) and circumscribed to (b); and dually, a triangle self-conjugate with regard to (b), and inscribed in (a). In either case, when there is one such triangle there are infinitely many.

A line pair or a point pair is apolar to another conic if its members are conjugate with regard to the latter. A line-double is apolar to a conic if the line touches the conic, and a point-double is apolar to a conic if the point lies on the conic.

We next define apolarity of quadrics.

DEFINITION. *Two quadrics*

$$\sum_{i,j=1}^{4} b_{ij}\, x_i\, x_j = 0, \tag{6}$$

$$\sum_{i,j=1}^{4} a_{ij}\, u_i\, u_j = 0 \tag{7}$$

that satisfy the relation

$$\sum_{i,j=1}^{4} a_{ij}\, b_{ij} = 0 \tag{8}$$

are said to be apolar to each other.

THEOREM. *Two apolar quadrics, neither of which is degenerate, have the property that there exist tetrahedra self-conjugate with regard to the one and inscribed in the other; and dually there exist tetrahedra self-conjugate with regard to the second, and circumscribed about the first.*†

This proof fails when either quadric is degenerate. We shall next determine

* Various other names are used for this relation. The word "apolar" is due to Reye, who also uses the expressions that (a) *supports* (stützt) (b), and that (b) *rests on* (ruht auf) (a). English writers call (a) *harmonically inscribed* in (b). Rosanes calls the two conics *conjugate*.

† For proof see Hesse's *Vorlesungen, Kap.* 16, p. 190.

the meaning of apolarity between a proper conic

$$\sum_{i,j=1}^{4} w_{ij}\, u_i\, u_j = 0, \qquad \Delta(w) = 0, \tag{9}$$

and an unrestricted point quadric

$$\sum_{i,j=1}^{4} a_{ij}\, x_i\, x_j = 0. \tag{10}$$

The relation is

$$\sum_{i,j=1}^{4} a_{ij}\, w_{ij} = 0, \tag{11}$$

and this is evidently an invariant for non-singular collineations of space. Hence there is no loss of generality if we take $x_4 = 0$ as the plane of the conic (w). Its equation then becomes

$$\sum_{i,j=1}^{3} w_{ij}\, u_i\, u_j = 0$$

and the relation of apolarity is

$$\sum_{i,j=1}^{3} a_{ij}\, w_{ij} = 0, \tag{12}$$

whence the

THEOREM. *The conic* (9) *is apolar to the quadric* (10) *if and only if it is apolar to the conic in which its plane cuts the quadric.*

Similarly it can be shown that if a point-pair is apolar to a quadric, its members are conjugate with regard to the quadric. For a point-double, the condition is that the point lie on the quadric.

Now it is clear that every quadric (w) whose coördinates satisfy the linear equations

$$\sum_{i,j=1}^{4} a_{ij}^{(k)}\, w_{ij} = 0 \qquad (k = 1, \cdots, r) \tag{13}$$

s apolar to each of the quadrics

$$\sum_{i,j=1}^{4} (\lambda_1\, a_{ij}^{(1)} + \cdots + \lambda_r\, a_{ij}^{(r)})\, x_i\, x_j = 0. \tag{14}$$

This system we shall call the *basal system* of the linear system of quadrics (w) defined by (13).

We shall denote the system of ∞^r conics (embedded in ∞^{r+1} quadrics) defined by a set of linear equations (13) of which $8 - r$ are linearly independent, by L_r, and we now proceed to study the systems L_r from $r = 7$ to $r = 0$.

1. The L_7 or seven-parameter family is defined by one equation

$$\sum_{i,j=1}^{4} a_{ij}\, w_{ij} = 0. \tag{15}$$

Then the basal system consists of a single quadric

$$\sum_{i,j=1}^{4} a_{ij}\, x_i\, x_j = 0, \tag{16}$$

and every conic of the family is apolar to this quadric.

We have then the result: *In the general L_7, the conics that lie in any plane are those which are apolar to a fixed conic in the plane, this fixed conic being the intersection of the plane with the base quadric* (16). *The system contains degenerate conics; in fact, the point-pairs of the general L_7 are ∞^5 in number. An arbitrary point of space may be associated with any one of the ∞^2 points of its polar plane in the base quadric to form a point-pair of the L_7. The point-doubles of the system consist of the points of the base quadric itself.*

The only planes, in general, in which the configuration is projectively different from that of the general plane are those tangent to the base quadric. In such a plane the base quadric cuts a pair of lines; and these are conjugate with regard to every conic of the L_7 in this plane.

Special types of L_7.

(*A*) The base quadric is a cone.

The system has then the following distinctive features:

(*a*) the ∞^2 special planes pass through the vertex; (*b*) there are ∞^1 of them, the tangents to the cone, in which all the ∞^4 conics of the L_7 touch a fixed line; (*c*) the vertex of the cone is singular, in that it can be associated with any point of space to form a point-pair of the L_7.

(*B*) The base quadric is a plane-pair.

In this case we have singular planes; namely the L_7 includes every conic in either of the planes of the basal plane-pair.

Conversely, if an L_7 includes all the conics of any plane, its basal quadric must be a plane-pair of which this plane is one member.

(*C*) The base quadric is a plane-double. Then (15) is of the form

$$\sum_{i,j=1}^{4} d_i\, d_j\, w_{ij} = 0.$$

We have studied this system (§ 3, 2) and have seen that it defines the quadrics that touch the plane (d); in particular, then, the conics of the system are those that touch (d), including those that lie in this plane. In S_9 this equation represents a hypersurface tangent to Σ at all points of a S_5.

2. The L_6, or 6-parameter family, is defined by two distinct equations

$$\sum_{i,j=1}^{4} a_{ij} w_{ij} = 0, \qquad \sum_{i,j=1}^{4} b_{ij} w_{ij} = 0. \tag{17}$$

The basal system consists of the pencil of quadrics

$$\sum_{i,j=1}^{4} (\lambda a_{ij} + \mu b_{ij}) x_i x_j = 0.$$

These have, in general, a common quartic curve C_4. Conversely, any conic apolar to two quadrics of the system (21) is apolar to all, and hence belongs to L_6.

The conics of the L_6 in an arbitrary plane are ∞^3 in number, and are apolar to all the conics of the pencil in which the basal system cuts the plane. The latter conics pass through the four points P_1, P_2, P_3, P_4, in which C_4 intersects the plane; and among them are in general three line-pairs. Each of these is a conjugate pair with regard to any conic of the L_6. Conversely, if a conic has two of these as conjugate line-pairs, then it is apolar to two distinct quadrics of the basal system, and hence belongs to the L_6.

This gives us the following

THEOREM. *Let a plane (π) cut C in the four points P_1, P_2, P_3, P_4. Then the conics of L_6 that lie in (π) are those that have these points as a "polar quadrangle."* (Cf. Reye, *Geometrie der Lage*, 4th ed., vol. 1, p. 261 ff.)

Degenerate conics of the L_6. The point-doubles are necessarily common to all the basal quadrics, and hence are the points of the C_4. The members of a point-pair must be conjugate with regard to each of the basal quadrics. It is known that the polar planes of a point P with regard to a pencil of quadrics meet in a line l_p. Hence P and any point of this line form a point-pair of the L_6.

There are, however, four points which are exceptional to the above statement, namely the vertices of the cones of the basal pencil. We have then the result:

THEOREM. *The point-doubles of L_6 are the ∞^1 points of the curve C_4 common to the basal quadrics. There are four of them in an arbitrary plane* (cf. § 3 at end). *An arbitrary point determines a line, any point of which may be associated with it to form a point-pair of the L_6. But there are four singular points, the vertices of the cones of the basal system, each of which can be so associated with any point of the plane of the other three.*

We have seen (§ 3) that the point-pairs whose members lie on a given line are represented in S_9 by a S_2^1; hence any S_7 contains one of them. Hence the

THEOREM. *The L_6 includes one point-pair whose members lie on an arbitrary line.*

Special types of L_6. In the foregoing, we have tacitly assumed that the pencil of basal quadrics is of the most general type. It is clear that the classification of L_6's is the same as that of pencils of quadrics. The latter is well known, and we shall not give it here.

We observe, however, that the first important specialization occurs when the pencil includes a plane-pair. In this case we may define the basal system by means of the plane-pair and any other member. Then the L_6 is the totality of members of an L_7 with regard to which two fixed planes are conjugate.

Similarly we get other special types. In particular, the totality of conics that touch two fixed planes form a special L_6.

THEOREM. *An L_6 can be such as to include all the conics of one plane, but not of more than one plane.*

For consider the L_6 whose basal system is defined by two plane-pairs having a common plane; every conic in this plane belongs to the L_6. To prove the second part of the theorem, we note that an L_6 can be defined as the totality of conics common to two L_7's. But there is one and only one L_7 that contains all the conics of two planes, namely the one whose basal quadric is this plane-pair. Hence there is no L_6 that includes all these conics.

3. The L_5 is defined by three independent equations

$$(18) \qquad \sum_{i,j=1}^{4} a_{ij}\, w_{ij} = 0, \qquad \sum_{i,j=1}^{4} b_{ij}\, w_{ij} = 0, \qquad \sum_{i,j=1}^{4} c_{ij}\, w_{ij} = 0.$$

The basal system is a *net* or linear two-parameter system represented by

$$\sum_{i,j=1}^{4} (\lambda a_{ij} + \mu b_{ij} + \nu c_{ij})\, x_i\, x_j = 0,$$

and the L_5 consists of all the conics apolar to every member of this system.

The basal system cuts an arbitrary plane in a so-called "bundle" or linear two-parameter family of conics. The ∞^2 conics of the L_5 in this plane are all apolar to the conics of the bundle, and they comprise a linear family in tangential coördinates. Reye (loc. cit.) calls such a family a "Schaarschaar." The nature of these two systems is well known. The conics of a bundle have in general no common point, those of a "Schaarschaar" no common tangent. A special case consists of the conics that have three common points in the one case, or in the other have three common tangents.*

Degenerate conics of the L_5. Clearly the point-doubles are eight in number, being the points common to all the base-quadrics. Thus we have (cf. § 3) the

THEOREM. *The point-doubles of space correspond to points of a S_3^8 lying on Σ.*

* For further treatment see *Encyklopädie*, III C 1, § 81.

There are also ∞^1 singular points, each of which may be associated with any point of a line to form a point-pair of the system. These singular points are the vertices of the cones of the basal system, and Reye (loc. cit.) shows that they lie on a sextic curve called the "Kerncurve" of the bundle. This curve has been considerably studied by Reye, and also, as an adjunct to a net of quadrics, by Sturm.*

Special types of L_5. There are a large number of special cases, and as before we shall not classify them. In general the basal net contains no plane-pair nor plane-double. If it contains one or more of either, the L_5 is correspondingly specialized. In particular, the conics that touch three fixed planes form a special L_5.

THEOREM. *An* L_5 *can contain all the conics of a plane. In this case it contains no other conics, and the conics of the* L_5 *in an arbitrary plane are the point-pairs on the line in which it intersects the plane of the* L_5.

4. The L_4 is a system of conics determined by four independent linear equations

$$\sum_{i,j=1}^{4} a_{ij}\, w_{ij} = 0, \qquad \sum_{i,j=1}^{4} c_{ij}\, w_{ij} = 0,$$

$$\sum_{i,j=1}^{4} b_{ij}\, w_{ij} = 0, \qquad \sum_{i,j=1}^{4} d_{ij}\, w_{ij} = 0. \tag{19}$$

The basal system of quadrics is given by

$$\sum_{i,j=1}^{4} (\lambda_1 a_{ij} + \lambda_2 b_{ij} + \lambda_3 c_{ij} + \lambda_4 d_{ij})\, x_i\, x_j = 0.$$

This system is called in German a *Gebüsch*. Its ∞^3 members have in general no common point. They include ∞^2 cones, and in general ten plane-pairs.†

The Gebüsch being completely defined by any four of its members that are linearly independent, we may define it by means of four of its ten plane-pairs. If four plane-pairs be chosen arbitrarily, there is a Gebüsch linearly dependent on them. Hence we may define the L_4 as follows:

THEOREM. *The general* L_4 *has the property that there are ten pairs of planes, each a conjugate pair with regard to every conic of the* L_4. *Given four arbitrary pairs of planes, the totality of conics with regard to which each pair are conjugate constitute an* L_4 *of general type; and there are six additional pairs of planes, each pair conjugate with regard to all the conics of the system.*

Degenerate conics. In general the L_4 has no point-doubles. It has ∞^2 point-pairs. To determine the order of the surface on which they lie, we recall (§ 3) that the point-pairs with one member on a given line are repre-

* See *Encyklopädie*, III C 2, p. 248, note 543.

† For proof see Reye, loc. cit., § 6.

sented in S_9 by a S_4^4. Hence in an arbitrary S_5 there are four points of this locus, and so in the general L_4 there are *four* point-pairs of which one member lies on an arbitrary line. Hence the points of the point-pairs lie on a quartic surface Φ_4.

THEOREM. *The surface Φ_4 whose existence has just been established is the locus of vertices of the ∞^2 cones of the basal system.*

We observe that the L_4 has singular planes, in each of which there are ∞^2 conics instead of ∞^1. Namely, in any plane of the basal plane-pairs the conics of the L_4 form a *Schaarschaar* of the type considered in 3.

Of course there are numerous special types of L_4. In particular, the conics that touch four fixed planes form a special L_4. Thus we have the following corollary to an earlier theorem:

The conics that touch four fixed planes form a system L_4, and there exist six other pairs of planes each conjugate with regard to every conic of the L_4.

5. The L_3, or three parameter system, is defined by five independent equations:

$$\sum_{i,j=1}^{4} a_{ij}^{(k)} w_{ij} = 0 \qquad (k = 1, \cdots, 5). \tag{20}$$

These equations define ∞^4 envelope-quadrics, each apolar to the ∞^4 point-quadrics of the basal system

$$\sum_{i,j=1}^{4} \left(\sum_{k=1}^{5} \lambda_k a_{ij} \right) x_i x_j = 0. \tag{21}$$

There is complete duality between these two linear systems.* The former includes ∞^3 conics, which comprise the L_3, and the latter includes a dual system of ∞^3 cones.

We know that the conditions that a quadric be a conic lying in a given plane (p) are the four linear equations

$$\begin{aligned} \sum_{i=1}^{4} w_{1i} p_i = 0, \qquad \sum_{i=1}^{4} w_{3i} p_i = 0, \\ \sum_{i=1}^{4} w_{2i} p_i = 0, \qquad \sum_{i=1}^{4} w_{4i} p_i = 0. \end{aligned} \tag{22}$$

These, together with the five equations (20), form a system that has in general one and essentially only one solution; hence *the L_3 has one conic in an arbitrary plane.*

THEOREM. *The conics of the L_3 that lie in planes through a fixed line generate a surface of order 8, having the fixed line as a sextuple line.*

Degenerate conics of the L_3. We may state the

* Cf. Reye, loc. cit., § 7.

THEOREM. *The point-pairs of the L_3 are ∞^1 in number, and lie on a curve C_{10} of order ten. Their axes generate a ruled surface K_{10} of order ten. Dually, the basal system includes ∞^1 plane-pairs, which envelope a developable Γ_{10} of class ten; their lines of intersection lie on a ruled surface Φ_{10} of order ten.*

Reye (loc. cit., § 7) gives a number of very interesting relations among these four figures.

6. The L_2, or two-parameter linear system, is defined by a set of six independent equations. The totality of sets of solutions of such a system are linearly dependent on four, and define ∞^3 envelope-quadrics, which include the conics of the L_2.

Hence we have as alternative definition of the L_2 the equations

$$w_{ij} = \lambda_1 a_{ij} + \lambda_2 b_{ij} + \lambda_3 c_{ij} + \lambda_4 d_{ij} \quad (i, j = 1, \cdots, 4), \tag{23}$$

$$\Delta(w) = 0. \tag{24}$$

The system of quadrics (23) is dual to the *Gebüsch* discussed in 4; and the basal system is dual to the system of envelope-quadrics there discussed. We have then at once the following facts:

The conics of L_2 lie in ∞^2 planes, one in each plane in general, and these planes envelope a surface Φ_4 of class four.

The L_2 includes in general ten point-pairs; their axes all lie on Φ_4. The basal system contains ∞^2 plane-pairs, whose planes envelope the surface Φ_4. In other words, the planes of these plane-pairs are the planes that contain conics of the L_2. Since they can be associated into pairs that belong to the basal system, we have the

THEOREM. *The planes of the conics of the L_2 can be associated in pairs in such a way that the members of each pair are mutually conjugate with regard to every conic of the L_2.*

The general S_3 in S_9 meets in eight points the S_6^8 which represents the conics through an arbitrary point; hence the

THEOREM. *The L_2 includes 8 conics through an arbitrary point of space.*

Special types of L_2. As in the earlier cases, when the basal system contains any plane-doubles, the conics of L_2 all touch every such plane; and in particular, the conics that touch six fixed planes form an L_2 of very special type. The dual system (viz., the Gebüsch of quadrics through six fixed points) has been somewhat studied.*

In general, the L_2 will have no singular plane containing more than one conic. There may, however, be such planes. There are four types, of which one reduces to a complete S_3, and the others are incomplete L_2's with singular planes. Namely:

* Cf. *Encyklopädie*, III C 2, § 141.

(1) One singular plane containing ∞^1 conics.

(2) Two singular planes, each containing ∞^1 conics.

(3) One singular plane containing ∞^2 conics.

(4) If all four conics are coplanar, every linear combination of them is a conic, and we have a complete three-parameter linear system of conics in the plane in question.

7. The L_1, or one-parameter linear system, is most conveniently defined as a system linearly dependent on three sets of numbers:

$$w_{ij} = \lambda_1 a_{ij} + \lambda_2 b_{ij} + \lambda_3 c_{ij}, \tag{25}$$

$$\Delta(w) = 0. \tag{26}$$

THEOREM. *The planes of the conics of a general L_1 form a developable of class* 6.

For the conics whose planes pass through a given point are represented in S_9 by a S_7^6 (§ 3) and hence an arbitrary S_2 contains 6 of them. We find easily also the following results.

THEOREM. *The conics of the general L_1 lie on a surface of order eight.*

THEOREM. *There are four conics of the L_1 that touch an arbitrary plane.*

COROLLARY. *The conics of the L_1 cut a given plane in points of a curve of order* 8. *They determine on this curve an involution of points, in which four pairs of points are self-corresponding.*

The basal system of the L_1 consists of a linear ∞^6 of point-quadrics. We know (2) that such a system includes eight plane-doubles. But every envelope-quadric touches every plane that is a plane-double apolar to it; so that every conic of the L_1 touches these eight fixed planes. This leads to another definition of the L_1.

THEOREM. *The general L_1 consists of the totality of conics that touch eight fixed planes, whereof seven may be chosen arbitrarily.*

In other words: *Let seven planes be chosen arbitrarily in space. There are ∞^1 planes, generating a sextic developable, in each of which the seven lines cut by the fixed planes are all tangent to a conic. The conics thus determined lie on a surface of order eight, and all touch an eighth fixed plane.*

8. The L_0, or system of conics whose coördinates satisfy *eight* linear equations, may be defined by

$$w_{ij} = \lambda_1 a_{ij} + \lambda_2 b_{ij} \qquad \Delta(w) = 0. \tag{27}$$

We may state the result:

The general L_0 consists of four conics having common tangent planes that generate a quartic developable. A system of this kind is uniquely determined by any two quadrics; the planes of the four conics form the only tetrahedron that is self-conjugate with regard to all the quadrics of the series containing the L_0.

7. Complete Linear Systems

We now propose to determine all possible types of complete linear systems. First we will note that such a system may be given parametrically in the form

$$w_{ij} = \lambda_1 a_{ij}^{(1)} + \cdots + \lambda_7 a_{ij}^{(7)} \qquad (i, j = 1, \cdots, 4),$$

where $\Delta(w) \equiv 0$ for all values of $(\lambda_1, \cdots, \lambda_7)$.

Consider, in particular, two conics (a) and (b) of a complete linear system. Then every quadric

$$w_{ij} = \lambda a_{ij} + \mu b_{ij}$$

must be a conic of the system. We have then

$$\lambda^4 \Delta(a) + \lambda^3 \mu \Theta(a, b) + \lambda^2 \mu^2 \Phi(a, b) + \lambda \mu^3 \Theta(b, a) + \mu^4 \Delta(b) = 0.$$

We have already, since (a) and (b) are conics,

$$\Delta(a) = \Delta(b) = 0,$$

and the following further relations are necessary:

$$\Theta(a, b) = 0, \qquad \Theta(b, a) = 0, \qquad \Phi(a, b) = 0.$$

From the results of § 5, it follows that we must have one of the following situations:

(*A*) If neither (a) nor (b) is degenerate, then (1) they are coplanar; (2) each touches the plane of the other, and they cut the line of intersection of their planes in a harmonic set of points.

(*B*) When (a) is a proper conic, and (b) is degenerate, then (1) if (b) is a point-pair, its axis meets (a), and the plane of (a) touches (b); (2) if (b) is a point-double, the plane of (a) touches it.

(*C*) When both conics are degenerate, then (1) if both are point-pairs, their axes intersect; (2) if either is a point-double, the equation is satisfied.

Considering these cases, we see that except in *A*2 and *B*1, the conics are coplanar. In case *A*2 they are necessarily tangent; and in case *B*1, either they are coplanar, or the point-pair (b) has one member lying on (a).

We introduce now the following

Definition. *A point-pair shall be said to be tangent to a proper conic, when one of its points lies on the conic.*

And we have the result: *Every two conics of a complete linear system are either coplanar or tangent. Conversely, we see without difficulty that if two conics are coplanar or tangent, every quadric linearly dependent on them is a conic.*

Concerning the locus of these conics, we present without detailed proof the following statement:

Theorem. *If* $(w) = \lambda(a) + \mu(b)$ *be a one-parameter complete linear system, it is of one of the two types*:

(1) *the* SERIES (*Schaar*) *consisting of the conics in a plane that touch four fixed lines in the plane*;

(2) *the one-parameter* SHEAF, *consisting of the conics cut from a quadric cone by the planes of a pencil whose axis is a tangent line to the cone. The axis of the pencil, which is evidently tangent to all the conics of the sheaf, we shall call the axis of the sheaf; and the point of contact we shall call the vertex of the sheaf.*

We may divide complete linear systems in general into two classes, according as all their members lie in the same plane or not. The planar systems we shall not discuss further than to remark that they are the systems studied in plane geometry as linear systems in tangential coördinates.

Let us consider a system in which not all the conics are coplanar. By brief reasoning, which we here suppress, one arrives at this result:

THEOREM. *In any non-planar complete linear system, the members all touch a fixed line at a fixed point; and conversely, any number of conics tangent to a fixed line at a fixed point will determine a complete linear system.*

We have not considered degenerate conics. Evidently a point-pair can belong to a system of this type if and only if it has either (1) one member at the vertex, or (2) both members on the axis; and a point-double can belong to the system, if and only if it lies on the axis.

DEFINITION. *A non-planar linear complete system, of which the coördinates of the members are linearly dependent on those of $r+1$ linearly independent conics, shall be called an r-parameter sheaf.*

We now examine the most general sheaf and find readily the

THEOREM. *The most general sheaf of conics consists of the totality of conics that touch a fixed axis at a fixed point. It is projectively equivalent to the system whose coördinates are given by the matrix*

$$\begin{bmatrix} 0 & 0 & 0 & a_1 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & a_0 & a_3 \\ a_1 & a_2 & a_3 & a_4 \end{bmatrix}$$

for all sets of values $(a_0, a_1, a_2, a_3, a_4)$ not all zero. It follows that such a sheaf is a four-parameter system.

DEFINITION. *A four-parameter sheaf shall be called an entire sheaf.*

We note the following properties of the entire sheaf:

An entire sheaf is determined by a point and a line through it, hence there are in all ∞^5 of them.

Any proper conic is a member of ∞^1 entire sheaves, one with its vertex at each point of the conic.

If two entire sheaves have in common a proper conic, their axes are coplanar, and they have in common a pencil of conics having double contact.

A point-pair belongs to ∞^2 entire sheaves, viz., to any one whose vertex is at either point of the pair, or whose axis is the axis of the pair.

Every two entire sheaves have in common one point-pair, made up of the vertices of the sheaves. Two entire sheaves have more than one common point-pair, if they have the same vertex, or if the vertex of one passes through the axis of the other.

A point-double belongs to every entire sheaf whose axis passes through the point.

Translating these results in S_9, we will denote by Λ the singular domain S_6^{10} that represents point-pairs, and by Φ the linear S_4 on Σ that represents an entire sheaf.

Then Σ contains ∞^5 Φ's. Through an ordinary point of Σ there are ∞^1, of which any two intersect in a line. In general, two Φ's do not intersect at any ordinary point of Σ, but at a single point of Λ. If they intersect at any ordinary point, they have a common line.

Through an arbitrary point of Λ, there are in general ∞^2 Φ's. Through any point of the S_3^3 that represents point-doubles there are ∞^3 Φ's.

The point-pairs of the above entire sheaf are determined by either $a_0 = 0$, or $a_1 = a_2 = 0$, corresponding respectively to point-pairs having one member at the vertex, and to those having both members on the axis. The former are represented in Φ by a linear three-space Q, the latter by a plane R.

The point-doubles are determined by $a_1 = a_2 = 0$, $a_0 a_4 = a_3^2$, and are represented in Φ by a S_1^2, i. e., a *conic*.

The entire sheaf contains sheaves of one, two, and three parameters, which we denote by T_1, T_2, T_3 respectively, with reference to which we give here only summary statements.

1. The one parameter sheaf T_1 we have already considered. If (a) and (b) be any two conics of T_4, then a T_1 is given by

$$w_i = \lambda a_i + \mu b_i \qquad (i = 0, \cdots, 4).$$

This corresponds to a *line* in S_4. There are therefore ∞^6 T_1's in any T_4.

If (a) and (b) are not coplanar, the conics lie on a cone that touches the axis at the vertex of T. In this case the sheaf contains just one degenerate conic. There are, however, special cases not described here.

2. The T_2 is defined by the equations

$$w_i = \lambda_1 a_i + \lambda_2 b_i + \lambda_3 c_i \qquad (i = 0, 1, 2, 3, 4),$$

where (a), (b), and (c) are linearly independent members of a T_4. It is represented in S_4 by a plane (π).

The plane (π) cuts Q in a line k and R in a point K, and can be described by lines drawn from K to the points of k. Each of these lines represents a planar T_1, as may easily be seen. Hence the T_2 may be regarded as composed of ∞^1 planar T_1's. Moreover, the common tangents to the members of each planar T_1 meet the axis at the same points R_1, R_2, since each of the lines cuts R at the same point. We have then the following definition of the T_2:

THEOREM. *The totality of conics of a T_4 that touch two fixed planes, whose intersection does not meet the axis of the T_4, constitute a T_2 of the most general type.*

Special types of T_2 correspond to special relations of (π) to the space containing the line k and the plane containing the point K. For example, it may happen that K lies on k. In this case *a T_2 of the type under consideration consists of all the conics of a T_4 that have contact of the second order with one another at the vertex, and also touch a fixed plane not through the vertex.*

Two other special types are evidently those in which either k or K is indeterminate. The latter is a planar T_2, and we find the

THEOREM. *The general planar T_2 consists of all the conics of a T_4 that lie in a plane and touch any two lines so drawn from two fixed points on the axis of the T_4 as to intersect on a fixed line in the plane. It follows that the common tangents to any two conics of the system intersect on this fixed line.*

3. The three-parameter system T_3 can be treated in the same way. A T_3 is represented in S_4 by a *space* S_3; and in general this cuts Q in a plane N, and R in a line n. To the former correspond ∞^2 point-pairs of the T_3, consisting of the vertex P, taken with the ∞^2 points of a plane (σ). To the latter correspond ∞^1 point-pairs on the axis of the T.

The S_3 may be regarded as swept out by a pencil of planes through the line n. Each of these planes corresponds to a planar T_2 of the kind just studied. We have seen that such a system has the property that the common tangents to any two of its members meet on a fixed line l. In our pencil of T_2's, these fixed lines all lie in the plane (σ). Also the vertex of the cone of any T_1 in the T_3 lies on (σ). We may then define the T_3 as follows:

Let a plane (σ) and a point-pair (R_1, R_2) on the axis be chosen arbitrarily. The totality of conics of the T_4, which touch a pair of lines drawn from R_1 and R_2 to any point of the plane (σ), generate a T_3. The common tangents to any two coplanar members of the T_3 meet on (σ), and the vertex of the cone through any two other members lies on (σ).

We mention two special types. There is one special T_3, namely, $a_0 = 0$, consisting of all the point-pairs whereof the vertex is one member. Another type is represented by a S_3 through the plane containing the point K. It is obvious that such a S_3 represents a planar T_3.

CAMBRIDGE, MASS.,
June, 1913.

THE WEIERSTRASS E-FUNCTION FOR PROBLEMS OF THE CALCULUS OF VARIATIONS IN SPACE*

BY

GILBERT AMES BLISS

For the problem of minimizing the definite integral

$$\int F(x, y, z, p, q, r)\, dt,$$

$$p = \frac{dx}{dt}, \qquad q = \frac{dy}{dt}, \qquad r = \frac{dz}{dt}, \tag{1}$$

where the function F has the homogeneity property

$$F(x, y, z, kp, kq, kr) = kF(x, y, z, p, q, r) \qquad (k > 0), \tag{2}$$

the Weierstrass E-function

$$E(x, y, z;\ p, q, r;\ p', q', r')$$

has the form

$$\begin{aligned} E = F(x, y, z, p', q', r') &- p' F_p(x, y, z, p, q, r) \\ &- q' F_q(x, y, z, p, q, r) \\ &- r' F_r(x, y, z, p, q, r). \end{aligned}$$

Behaghel† has deduced a very useful expression for the E-function in terms of the quadratic form

$$\begin{aligned} &Q(x, y, z;\ p, q, r;\ \xi, \eta, \zeta) \\ &\quad = F_{pp}\,\xi^2 + F_{qq}\,\eta^2 + F_{rr}\,\zeta^2 + 2F_{qr}\,\eta\zeta + 2F_{rp}\,\zeta\xi + 2F_{pq}\,\xi\eta. \end{aligned} \tag{3}$$

In his proof he makes use of functions, in his notation A, B, C, which are analogous to one which is used for a similar problem in the plane. For the space problem, unfortunately, these functions may become infinite even for very simple integrals of the type (1), and in such cases his proof of the formula would not be valid.‡

* Presented to the Society, December 26, 1913.

† Mathematische Annalen, vol. 73 (1913), p. 596.

‡ For the function

$$F = \sqrt{p^2 + q^2 + (q + r)^2}$$

the expression for A is

$$A = -F_{qr}/qr = \left(1 - \frac{p^2}{qr}\right) / [\,p^2 - q^2 + (q + r)^2\,]^{3/2},$$

and this is infinite for any direction in the y-plane or z-plane not perpendicular to the x-axis.

In the following sections it is proposed to give a proof of a formula somewhat more elegant than that of Behaghel and which can be reduced to his by a simple transformation. The objections mentioned above are avoided, and the special case when the directions (p, q, r) and (p', q', r') are in the same line but opposite, treated separately in Behaghel's paper, is here quite unexceptional.* The method used applies at once to problems in space of any number of dimensions. In § 2 some useful consequences of the formula are deduced, and in § 3 some peculiarities of the extension of the results found to spaces of higher dimensions are explained.

1. The Formula for the E-function

It will be supposed that for a fixed set of values (x, y, z, p, q, r) the function F has continuous first and second derivatives for arguments (p', q', r') defining directions in a cone shaped region P_ϵ, shown in Fig. 1

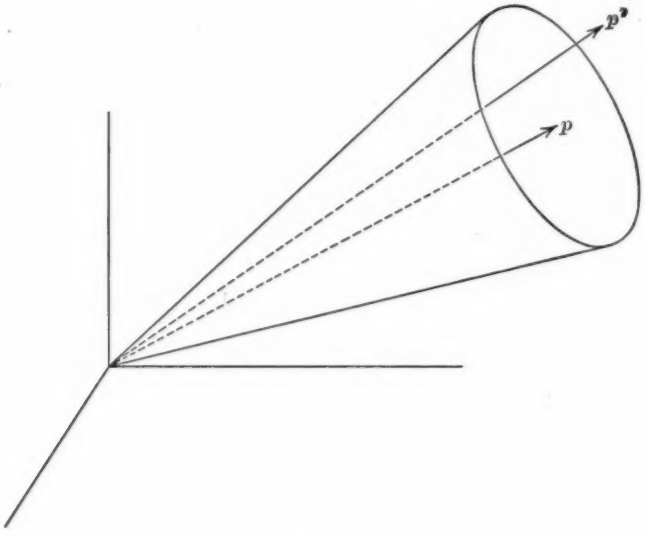

Fig. 1.

about an initial direction p, q, r. This and other directions will be referred to hereafter by their first letters only, as illustrated in the figure. Analytically the region is defined by an inequality of the form

$$|\omega| < \epsilon \tag{4}$$

* A formula given by Mason and Bliss, these Transactions, vol. 9 (1908), p. 459, is less effective than the one proved in the text for a similar reason. It does not necessarily hold when the two directions x', y', z' and x_u, y_u, z_u are opposite.

where ω is the angle between p and p' defined by the formula

$$\cos\omega = \frac{pp' + qq' + rr'}{\sqrt{p^2 + q^2 + r^2}\sqrt{p'^2 + q'^2 + r'^2}} \qquad (0 \leqq \omega \leqq \pi).$$

In as much as the properties of the E-function to be considered in this section do not depend upon the arguments x, y, z, the latter may be omitted in most of the equations. It is understood that for the direction p the quantities p, q, r are not all zero, and a similar remark applies for other directions.

The function F satisfies the relation

$$pF_p + qF_q + rF_r = F, \tag{5}$$

found by differentiating (2) for k and putting $k = 1$, and also the relations

$$\begin{aligned} pF_{pp} + qF_{pq} + rF_{pr} &= 0, \\ pF_{qp} + qF_{qq} + rF_{qr} &= 0, \\ pF_{rp} + qF_{rq} + rF_{rr} &= 0, \end{aligned} \tag{6}$$

which follow by differentiating (5) for p, q, r, respectively. Furthermore the first derivatives of F are positively homogeneous with order zero. For by differentiating (2) with respect to p, for example, it is found that

$$F_p(x, y, z, kp, kq, kr) = F_p(x, y, z, p, q, r) \qquad (k > 0).$$

From these results it follows readily that the E-function itself has the homogeneity property

$$E(x, y, z; kp, kq, kr; k'p', k'q', k'r') = k'E(x, y, z; p, q, r; p', q', r') \qquad (k > 0, k' > 0),$$

and the direction ratios which occur among its arguments may be taken without loss of generality to be direction cosines.

It will be helpful to use the notation

$$\begin{aligned} \Phi(p, \xi, \xi') = {} & \xi'(F_{pp}\xi + F_{pq}\eta + F_{pr}\zeta) \\ & + \eta'(F_{qp}\xi + F_{qq}\eta + F_{qr}\zeta) \\ & + \zeta'(F_{rp}\xi + F_{rq}\eta + F_{rr}\zeta). \end{aligned}$$

The form Φ has the following properties to be used later, the first of which gives an expression for the quadratic form Q in terms of Φ:

$$Q(x, y, z; p, q, r; \xi, \eta, \zeta) = \Phi(p, \xi, \xi), \tag{7}$$

$$\Phi(p, \xi, \xi') = \Phi(p, \xi', \xi), \tag{8}$$

$$\Phi(p, \xi, u\xi' + v\xi'') = u\Phi(p, \xi, \xi') + v\Phi(p, \xi, \xi''), \tag{9}$$

$$\Phi(p, \xi, p) = 0, \tag{10}$$

where in the third formula $u\xi' + v\xi''$ is a symbol for the values

$$u\xi' + v\xi'', \qquad u\eta' + v\eta'', \qquad u\zeta' + v\zeta''.$$

The first three of these formulas follow readily from the definition of Φ, while the last is a consequence of the homogeneity properties (6).

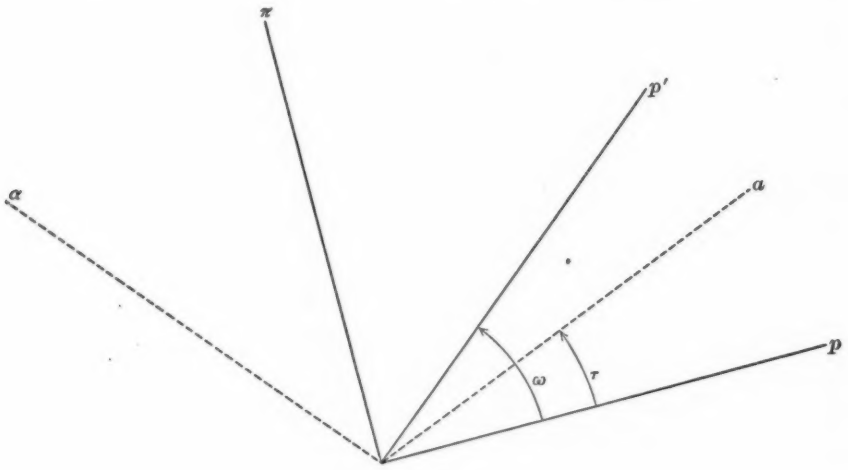

FIG. 2.

Consider now the two directions p and p' indicated in Fig. 2. There will always be a direction π which is orthogonal to p and co-planar with p and p'. Let a and α be two directions orthogonal to each other in the same plane with p, p', and π. Their direction cosines are expressible in the form

$$\begin{aligned} a &= p\cos\tau + \pi\sin\tau, & \alpha &= -p\sin\tau + \pi\cos\tau, \\ b &= q\cos\tau + \kappa\sin\tau, & \beta &= -q\sin\tau + \kappa\cos\tau, \\ c &= r\cos\tau + \rho\sin\tau, & \gamma &= -r\sin\tau + \rho\cos\tau, \end{aligned} \tag{11}$$

where τ is the angle between a and p. By changing the sense of the direction π, if necessary, it can be effected that for $\tau = \omega$ the direction a coincides with p'.

The E-function has then the expressions

$$\begin{aligned} E(p, p') &= \int_0^\omega \frac{d}{d\tau}\{p' F_p(a) + q' F_q(a) + r' F_r(a)\}d\tau \\ &= \int_0^\omega \Phi(a, \alpha, p')\, d\tau, \end{aligned}$$

which follow readily from (5), (11), and the definition of Φ. From the relations

$$p' = a\cos(\omega - \tau) + \alpha\sin(\omega - \tau),$$
$$q' = b\cos(\omega - \tau) + \beta\sin(\omega - \tau),$$
$$r' = c\cos(\omega - \tau) + \gamma\sin(\omega - \tau),$$

and the formulas (9) and (10) it is seen that

$$E(p, p') = \int_0^\omega \sin(\omega - \tau)\Phi(a, \alpha, \alpha)\,d\tau,$$

or, by applying the mean value theorem for a definite integral,

$$E(p, p') = (1 - \cos\omega)\Phi(a^*, \alpha^*, \alpha^*), \tag{12}$$

where a^* is the direction a corresponding to a suitably chosen value τ^* between 0 and ω, and α^* is the corresponding direction perpendicular to a^*.

Hence the function

$$E(x, y, z;\ p, q, r;\ p', q', r')$$

is expressible in the form

$$E = (1 - \cos\omega)Q(x, y, z;\ a, b, c;\ \alpha, \beta, \gamma). \tag{13}$$

In this formula ω is the angle $(0 \leqq \omega \leqq \pi)$ between the directions (p, q, r) and (p', q', r'), while (a, b, c) and (α, β, γ) are directions orthogonal to each other in the plane of the other two and defined by the formulas (11) *for a suitably chosen intermediate value of τ satisfying the inequality $0 < \tau < \omega$.* The relations between these directions is clearly shown in Fig. 2.

It is easy to derive Behaghel's formula from this result. The relations (11) give

$$p = a\cos\tau - \alpha\sin\tau, \qquad q = b\cos\tau - \beta\sin\tau, \qquad r = c\cos\tau - \gamma\sin\tau,$$

and with the help of (9), (10), and (8) it follows that

$$\Phi(a, p, p) = \sin^2\tau\,\Phi(a, \alpha, \alpha).$$

Hence the expression (12) can be put into the form

$$E(p, p') = (1 - \cos\omega)\frac{\Phi(a^*, p, p)}{\sin^2\tau^*}$$
$$= (1 - \cos\omega)\frac{Q(x, y, z;\ a^*, b^*, c^*;\ p, q, r)}{\sin^2\tau^*},$$

which is the formula required.

2. Consequences of the Formula

Consider an arc C defined by the continuous functions

$$(C) \qquad x = x(t), \qquad y = y(t), \qquad z = z(t) \qquad (t_1 \leqq t \leqq t_2)$$

and having a continuously turning tangent whose direction cosines will be represented by $p(t)$, $q(t)$, $r(t)$. The neighborhood C_ϵ of order zero of this arc is the totality of values (x, y, z, p, q, r) for each of which the inequalities

$$(C_\epsilon) \qquad |x - x(t)| < \epsilon, \quad |y - y(t)| < \epsilon, \quad |z - z(t)| < \epsilon, \quad (p, q, r) \neq (0, 0, 0)$$

are true for at least one value of t between t_1 and t_2. The neighborhood C'_ϵ of order one is similarly defined by the conditions

$$(C'_\epsilon) \qquad |x - x(t)| < \epsilon, \quad |y - y(t)| < \epsilon, \quad |z - z(t)| < \epsilon, \quad (p, q, r) \neq (0, 0, 0), \quad |\theta| < \epsilon,$$

where θ is the angle $(0 \leqq \theta \leqq \pi)$ between the directions p and $p(t)$. Besides the region P_ϵ of § 1 and the two neighborhoods just defined, there is a fourth region involved in the following theorems and defined by the conditions

(R) (x, y, z) *in a continuum of xyz-points;* $(p, q, r) \neq (0, 0, 0)$.

For each of the theorems the function F is supposed to be continuous and to have continuous first and second derivatives with respect to p, q, r in some region including the one involved in the statement of the theorem.

If for a fixed set of values (x, y, z, p, q, r) *the condition*

$$(14) \qquad E(x, y, z; p, q, r; p', q', r') \geqq 0$$

is satisfied for all directions (p', q', r') *in a neighborhood* P_ϵ *of the direction* (p, q, r), *then the condition*

$$(15) \qquad Q(x, y, z; p, q, r; \xi, \eta, \zeta) \geqq 0$$

is also satisfied for any values whatsoever of ξ, η, ζ. *In other words the necessary condition of Weierstrass, for either a strong or a weak minimum, at any point of a given curve, implies the necessity of Legendre's condition also.*

If in the formula (13) the direction p' approaches the direction p while remaining in a fixed plane through the latter, the directions a and α approach, respectively, p and a direction π normal to p in the plane. Hence the inequality (15) must be true when the direction ξ coincides with π. But p' may be made to approach p in any plane through p, and consequently π may be any normal to p. If ξ lies in a plane with p and π and makes an angle τ with p, the formulas

$$\xi = p \cos \tau + \pi \sin \tau, \quad \eta = q \cos \tau + \kappa \sin \tau, \quad \zeta = r \cos \tau + \rho \sin \tau,$$

together with (9) and (10) show that

$$Q(x, y, z;\ p, q, r;\ \xi, \eta, \zeta) = \sin^2\tau\, Q(x, y, z;\ p, q, r;\ \pi, \kappa, \rho).$$

Hence the inequality (15) is true for all values of ξ, η, ζ.

The form Q in the expression (13) is said to be regular in a region of values $(x, y, z;\ p, q, r)$ if it does not take opposite signs for different values of its arguments and vanishes only when ξ, η, ζ are all zero, or when the directions (ξ, η, ζ) and (p, q, r) are in the same line.

If the function Q is regular at all values (x, y, z, p, q, r) on an arc C, then there exists a neighborhood C'_δ in which it remains regular and does not take opposite signs.

For Q can always be transformed by a homogeneous orthogonal linear transformation of ξ, η, ζ* into a sum of squares whose coefficients $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation

$$\begin{vmatrix} F_{pp}+\lambda & F_{pq} & F_{pr} \\ F_{qp} & F_{qq}+\lambda & F_{qr} \\ F_{rp} & F_{rq} & F_{rr}+\lambda \end{vmatrix} = 0. \tag{16}$$

The term independent of λ in this equation is zero since it is the determinant of the coefficients of p, q, r in the equations (6). Hence one of the roots, say λ_3, is always zero; and the other two are different from zero and of the same sign for any set of values (x, y, z, p, q, r) corresponding to a point of the arc C, since for such values Q is regular and can vanish for one direction ξ, η, ζ only. Since the roots λ_1 and λ_2 are continuous functions of the arguments x, y, z, p, q, r, they will remain different from zero and will not change sign in a sufficiently small neighborhood C'_δ.

Suppose that Q is regular along the arc C and that the inequality

$$E[x(t), y(t), z(t);\ p(t), q(t), r(t);\ p', q', r'] > 0$$

is satisfied at every point of C whenever the direction (p', q', r') is distinct from that of C. Then there exists a neighborhood C'_ϵ such that in the region of values $(x, y, z;\ p, q, r;\ p', q', r')$ defined by the relations

$$(x, y, z, p, q, r) \text{ in } C'_\epsilon, \qquad (p', q', r') \neq (0, 0, 0)$$

the function E does not take opposite signs and vanishes only when the directions (p, q, r) and (p', q', r') coincide.

To prove this consider the function E_1 defined by the equations

$$\begin{aligned} &E_1(x, y, z;\ p, q, r;\ \pi, \kappa, \rho;\ \omega) = E/(1-\cos\omega) \qquad (0 < \omega \leqq \pi), \\ &E_1(x, y, z;\ p, q, r;\ \pi, \kappa, \rho;\ 0) = Q(x, y, z;\ p, q, r;\ \pi, \kappa, \rho), \end{aligned} \tag{17}$$

* See Kowalewski, *Einfuhrung in die Determinantentheorie*, § 116.

the first of which is formed by replacing the values (p', q', r') in E by expressions similar to (11) with $\tau = \omega$. From its definition E_1 is continuous for all values of its arguments for which the point (x, y, z) is sufficiently near to C, and $0 < \omega \leqq \pi^r$,* while the other arguments are the direction cosines of two perpendicular directions. It is continuous also for $\omega = 0$, since the formula (13) shows that as ω approaches zero the values of the quotient in the first part of the definition of E_1 approach continuously the values defined by the second of equations (17). Along the arc C the quotient E_1 is different from zero for directions π normal to $p(t)$, and for $0 \leqq \omega \leqq \pi^r$, on account of the hypothesis of the theorem and the expressions (17). Since it is continuous it retains these properties when the values (x, y, z, p, q, r) are in a neighborhood C'_ϵ of the arc C. The function E has therefore the properties described in the theorem, since E is expressible as a non-vanishing factor times E_1 whenever p and p' are distinct.

Consider a problem which is regular in a region R of the kind described in the first paragraph of this section, that is, a problem such that the form Q is everywhere regular in R. Then the function $E(x, y, z; p, q, r; p', q', r')$ does not take opposite signs when (x, y, z, p, q, r) and (x, y, z, p', q', r') are in R, and vanishes only when the directions (p, q, r) and (p', q', r') are coincident.

At every point of the region R the quadratic form Q is regular, and hence the expression (13) for the E-function vanishes only when p and p' coincide. Further any two sets of values $(x, y, z; p, q, r; p', q', r')$ can be joined by a continuous path over which p and p' are nowhere coincident. Hence the sign of the E-function is always the same at points where it does not vanish.

The regularity or non-regularity of the quadratic form Q is controlled by the behavior of a function F_1 defined by the equations

$$p^2 F_1 = F_{qq} F_{rr} - F_{qr}^2, \qquad qrF_1 = F_{pq} F_{pr} - F_{pp} F_{qr},$$

$$q^2 F_1 = F_{rr} F_{pp} - F_{rp}^2, \qquad rpF_1 = F_{qr} F_{qp} - F_{qq} F_{rp},$$

$$r^2 F_1 = F_{pp} F_{qq} - F_{pq}^2, \qquad pqF_1 = F_{rp} F_{rq} - F_{rr} F_{pq}.$$

The existence and continuity of this function are consequences of the relations (6). The coefficient of the first power of λ in the equation (16) is precisely F_1, and consequently λ_1 and λ_2 will be different from zero and have the same sign if and only if F_1 is positive.

A necessary and sufficient condition that the quadratic form Q be regular at a set of values (x, y, z, p, q, r) is

$$F_1(x, y, z, p, q, r) > 0.$$

* The symbol π^r stands for π radians.

3. Extension of the Preceding Results to Higher Spaces

The formula (13) which was the goal of the discussion of § 1 has an analogue for an integral of the form

$$\int F(y, p)\, dt,$$

where y and p are symbols for sets of elements of the form

$$y = (y_1, y_2, \cdots, y_n),$$

$$p = (p_1, p_2, \cdots, p_n) = \left(\frac{dy_1}{dt}, \frac{dy_2}{dt}, \cdots, \frac{dy_n}{dt}\right).$$

The function F is supposed to have a homogeneity property with respect to the arguments p similar to (2), and relations analogous to (5) and (6) are readily derived.

The E-function and the quadratic form Q have the forms

$$E(y, p, p') = F(y, p') - \sum_{i=1}^{n} p_i F_i,$$

$$Q(y, p, \xi) = \sum_{i,j=1}^{n} F_{ij}(y, p)\, \xi_i\, \xi_j$$

where F_i and F_{ij} are symbols for the derivatives $\partial F/\partial p_i$ and $\partial^2 F/\partial p_i\, p_j$, respectively. The E-function has the homogeneity property

$$E(y, \kappa p, \kappa' p') = \kappa' E(y, p, p') \qquad (\kappa > 0,\ \kappa' > 0),$$

and as before there will be no loss of generality if all the directions p are supposed to be normed, that is, to satisfy the relation

$$p_1^2 + p_2^2 + \cdots + p_n^2 = 1.$$

The condition for the orthogonality of two directions, p and π, in n-dimensional space is

$$p_1 \pi_1 + p_2 \pi_2 + \cdots + p_n \pi_n = 0.$$

The auxiliary form Φ has the definition

$$\Phi(p, \xi, \xi') = \sum_{i,j=1}^{n} F_{ij}(y, p)\, \xi_i\, \xi_j'$$

and satisfies four relations analogous to (7)–(10) of § 1.

There will always be at least $n - 2$ independent directions $q', q'', \cdots, q^{(n-2)}$ orthogonal to both p and p', which with p itself form a system of $n - 1$ independent directions; and the last system of $n - 1$ directions determines a unique direction π orthogonal to each of them. The directions π and p are independent, and orthogonal to each of the directions $q', q'', \cdots, q^{(n-2)}$. Since every other direction having this property must be expressible linearly

in terms of p and π, it follows that p' must have the form

$$p_i' = p_i \cos\omega + \pi_i \sin\omega \quad (i = 1, 2, \cdots, n\,;\, 0 \leqq \omega \leqq \pi').$$

In order to satisfy the last restriction in the parenthesis it may be necessary to change the sign of the direction π.

If the directions a and α are defined by the equations

$$a_i = p_i \cos\tau + \pi_i \sin\tau, \qquad \alpha_i = -p_i \sin\tau + \pi_i \cos\tau \quad (i = 1, 2, \cdots, n),$$

the argument proceeds exactly as in § 1, and the formula found for the E-function is

$$E(y, p, p') = (1 - \cos\omega)\, Q(y, a, \alpha),$$

where a and α are defined by the equations just given for a suitably chosen intermediate value of τ between 0 and ω.

All of the results of § 2 except the last can be proved for n dimensions in a manner analogous to that which is effective for three dimensions. The analogue of F_1 is a function defined by relations of the form

$$p_i\, p_k\, F_1 = A_{ik} \qquad (i, k = 1, 2, \cdots, n),$$

where A_{ik} is the cofactor of F_{ik} in the matrix of these derivatives. In the equation corresponding to (16) for n dimensions the term independent of λ vanishes as before, and F_1 is the coefficient of the first power of λ. Hence if Q is regular at a set of values (y, p) it will remain so in any neighborhood of those values in which F_1 is different from zero. But more than the requirement that F_1 shall have a certain sign is needed in order to insure the regularity of Q.

THE UNIVERSITY OF CHICAGO.

THE SUBGROUPS OF THE QUATERNARY ABELIAN LINEAR GROUP*

BY

HOWARD H. MITCHELL

1. Introduction

With the possible exception of the binary modular groups there is no finite linear group which is of more importance in analysis and geometry than what is known as the *abelian*† linear group. In the investigations of Hermite and Jordan the coefficients were taken as rational integers reduced modulo p (a prime), in which case it is the Galois group of the equation for the p-section of the periods of hyperelliptic functions.‡ It is also the group of isomorphisms of a certain type of commutative group and is isomorphic with certain systems of collineation groups.§

For the case of four variables and the modulus 3, it is the group of the equation for the 27 straight lines on a general cubic surface‖ as well as the group for the problem of reducing a binary sextic to the canonical form T^2-U^3. One of the collineation groups with which it is isomorphic in this case is the quaternary G_{25920}, which has received considerable attention.

If the coefficients of the transformations, instead of being integers reduced modulo p, are taken as marks of a general Galois field $GF(p^n)$, a more general group is obtained which has been investigated by Dickson.¶ This group requires an even number of variables and has a certain invariant bilinear function analogous to that for the ordinary continuous group in space which

* Presented to the Society at New York, April 25, 1914.

† This group is called abelian on account of its connection with abelian functions. It is not commutative.

‡ Cf. Jordan, *Traité des substitutions*, pp. 171–186; 354–369; Witting, Mathematische Annalen, vol. 29 (1886), p. 157; Dissertation (Göttingen, 1887); Burkhardt, Mathematische Annalen, vol. 38 (1891), p. 163.

§ Cf. Jordan, loc. cit., pp. 420–450; Witting, loc. cit.; Burkhardt, loc. cit.; Dickson, these Transactions, vol. 1 (1900), pp. 30–38; the author, Bulletin of the American Mathematical Society, vol. 20 (1913), pp. 134–138.

‖ Cf. Jordan, loc. cit., pp. 316–319; Dickson, *Linear Groups*, pp. 303–307; Coble, Johns Hopkins University Circular (1908) no. 7, pp. 80–88.

¶ Quarterly Journal of Mathematics, vol. 29 (1897), pp. 169–178, vol. 31 (1899), pp. 383–4; *Linear Groups*, pp. 89–109.

has an invariant linear complex. It is called *general* or *special* according as this function is left relatively (to within a factor in the field) or absolutely invariant. The first has an invariant subgroup of order $p^n - 1$ consisting of all the transformations which multiply each variable by the same mark of the field, and the second has for $p > 2$ an invariant subgroup of order 2 which contains a transformation changing the sign of each variable. The quotient group of the second with respect to its invariant subgroup of order 2, Dickson has denoted by the symbol, $A(2m, p^n)$, $2m$ representing the number of variables. For $p = 2$ the *special* group is itself represented by this symbol. These groups are simple except for $m = 1$, $p^n = 2, 3$ and $m = 2$, $p^n = 2$.

The object of this paper is the determination of the maximal subgroups of the $A(2m, p^n)$ for $m = 2$ and $p > 2$, although incidentally some other subgroups are noticed. For this purpose we consider the collineation group in a modular space* having an invariant linear complex. The group of all such collineations is $(1, 1)$ isomorphic with the quotient group of the *general* linear group with respect to its invariant subgroup of order $p^n - 1$. For $p > 2$ it contains an invariant subgroup of index 2, each transformation in which multiplies the linear function of the plücker line coördinates by a square in the field. This multiplier may be made unity by multiplying each coefficient of the collineation by the same properly chosen factor. This group is thus $(1, 1)$ isomorphic with the quotient group of the *special* linear group with respect to its invariant subgroup of order 2, i. e. with Dickson's $A(4, p^n)$. We shall depart slightly from Dickson's notation and shall represent the whole collineation group by $A_\nu(p^n)$ and its invariant subgroup by $A_1(p^n)$.

Although considerable work has been done on the determination of the subgroups, the problem seems to have been completely solved only for $n = 1$, $p = 3$.† For the general case Dickson has found a complete list of conjugate operators,‡ and has discussed a number of the subgroups.§ He has also shown that for $n = 1$, $p > 3$ there is no subgroup of index less than

$$p^3 + p^2 + p + 1.\|$$

The author has obtained a complete set of maximal subgroups in the general case. The possible subgroups whose orders are prime to the modulus p were

* Cf. Veblen and Bussey, these Transactions, vol. 7 (1906), pp. 241–259.

† Dickson, these Transactions, vol. 5 (1904), pp. 126–166. Jordan had previously shown that there was no subgroup of index less than 27 (loc. cit., pp. 319–329).

‡ These Transactions, vol. 2 (1901), pp. 103–138.

§ These Transactions, vol. 1 (1900), pp. 91–96; vol. 4 (1903), pp. 371–386; vol. 5 (1904), pp. 1–38; Quarterly Journal of Mathematics, vol. 32 (1900), pp. 42–63; Bulletin of the American Mathematical Society, vol. 10 (1903), pp. 178–184.

‖ These Transactions, vol. 6 (1905), pp. 48–57. Cf. also Jordan, loc. cit., p. 666, note *E*.

determined in a previous paper.* They were found to be the same as the ordinary finite groups which have an invariant linear complex.

With the exception of a few groups which exist only for particular moduli the subgroups whose orders are divisible by p are found to be of two general types. Those of the first type are the $A_1(p^k)$, if k is a factor of n, and the $A_\nu(p^k)$, if $2k$ is a factor of n. Those of the second type are analogous to the ordinary continuous groups which have an invariant linear complex. They have an invariant point and plane, a congruence (three types), a quadric (two types), or a twisted cubic.

As a check on our list of maximal subgroups we find that they agree with Dickson's† for the special case $n = 1$, $p = 3$. For this case he found them to be of indices 27, 36, 40, 40, 45. We also find that for $p^n > 3$ there is no subgroup of index less than $p^{3n} + p^{2n} + p^n + 1$, a result due to him‡ for $n = 1$.

2. General Properties of the Group

There are three types of transformations of period p in $A_1(p^n)$ if $p > 3$ and two types if $p = 3$. The first, which we shall refer to as *central elations*, leave fixed all the points in a plane and all the planes and lines through a point in that plane. All the lines in the plane of fixed points which pass through the point of fixed planes belong to the invariant complex.

The transformations of the second type leave invariant all the points on a line and every line of a flat pencil through each point of the axis. We shall refer to them as *skew elations*. The axis belongs to the complex as do also two of the above mentioned pencils. The coördinates of these pencils need not be in the $GF(p^n)$ however.

The transformations of the third type have a single invariant point, an invariant line through that point and an invariant plane through that line. All the lines in the fixed plane which pass through the fixed point belong to the complex. The path curves are conics in the fixed plane and twisted cubics outside of that plane. If $p = 3$ these transformations are of period 9.

There are two types of transformations of period 2 in $A_1(p^n)$.§ Both of them leave invariant all the points on each of two skew lines, but in the one case these lines are lines of the complex and in the other they are axes of a congruence contained by the complex. In the former case the coördinates of these lines are or are not in the $GF(p^n)$ according as p^n has the form $4l + 1$ or $4l - 1$. In the latter case their coördinates are always in the $GF(p^n)$.

* These Transactions, vol. 14 (1913), pp. 123–142. This paper will be referred to as Transactions I.

† Loc. cit.

‡ Loc. cit.

§ For a more general result see Dickson, *Linear Groups*, § 289; Annals of Mathematics, ser. 2, vol. 6 (1905), p. 148.

3. Groups Containing Central Elations

We proceed now to prove

Theorem 1. *The only subgroups of $A_1(p^n)$ which contain central elations are:*

(1) *groups having an invariant point and plane with coördinates in the $GF(p^n)$;*

(2) *groups having an invariant line of the complex with coördinates in the $GF(p^n)$;*

(3) *groups having an invariant congruence hyperbolic with respect to the $GF(p^n)$;*

(4) *groups of the same type and order as $A_1(p^k)$, where k is a factor of n;*

(5) *groups of the same type and order as $A_r(p^k)$, where $2k$ is a factor of n.*

We consider a particular subgroup of $A_1(p^n)$ and suppose that the largest subgroup it contains which consists of central elations with common center and axial plane is of order p^k. We denote such a group by (E_1). If all the other centers of elations are in the axial plane of (E_1), either that plane or a line in that plane through the center of (E_1) must remain invariant. If this is not the case we may suppose the existence of a second group of elations (E_2) with center not in the axial plane of (E_1).

These two additive groups leave invariant in common the line joining their centers and the line of intersection of their axial planes. The group generated cannot by hypothesis contain an additive group of elations of higher order than p^k. Hence its order on the line joining the centers must be

$$\tfrac{1}{2}(p^k+1)p^k(p^k-1)$$

or $60\,(p^k=3)$.* The group itself is of twice this order, since it contains an invariant reflection whose axes are axes of the invariant congruence. We shall first assume either that $p^k>3$ or that if $p^k=3$ no group of order $2\cdot 60$ of this type is present. The exceptional case will be considered later.

In the general case, if the invariant complex be $p_{14}+p_{23}=0$, every group of the sort described may be shown to be conjugate under $A_r(p^n)$ with that generated by

$$(E_1):[x_1, x_2+\lambda x_3, x_3, x_4], \qquad (E_2):[x_1, x_2, \lambda x_2+x_3, x_4],$$

where λ takes all values in the $GF(p^k)$.

Either the invariant congruence of this group must remain fixed or else there will be elations with centers not on either of the two axes. We denote one such group of elations by (E_3). The line through its center which is in the congruence is a line of the complex and hence must be in the axial plane of (E_3).

* The binary modular groups for the general Galois field were first determined by E. H. Moore and Wiman. For a new determination of these groups and for references to the earlier work the reader may consult a paper by the author in these Transactions, vol. 12 (1911), pp. 207–211.

The center of (E_3) must be in one of the $p^k + 1$ planes conjugate with the axial planes of (E_1) and (E_2) under the group generated by them. For the group (E_3) cannot be commutative with both (E_1) and (E_2) and hence we may suppose that it is not commutative with (E_2). Hence (E_2) and (E_3) must generate a group of the same sort as the above and this will contain an invariant reflection. This reflection and the one commutative with (E_1) and (E_2) must generate a dihedral group of order $2p$ under which the reflections are conjugate. If the center of (E_3) does not lie in one of the $p^k + 1$ planes mentioned above, its center will be conjugate under this group with a point on the line joining the centers of (E_1) and (E_2), which is distinct from any of the $p^k + 1$ centers on that line. But this is impossible. Hence the center of (E_3) must be in one of the $p^k + 1$ planes and we may suppose that it is in the axial plane of (E_1).

Since the group generated by (E_1) and (E_2) is invariant under all transformations of the form

$$[\alpha x_1 + \beta x_4, x_2, x_3, \gamma x_1 + \delta x_4],$$

where $\alpha, \beta, \gamma, \delta$ are any marks of the $GF(p^n)$ such that $\alpha\delta - \beta\gamma = 1$, it follows that we may choose the center of (E_3) as $(1\,1\,0\,0)$. In order that (E_2) and (E_3) shall generate a group of order $(p^k + 1)p^k(p^k - 1)$ the latter must be given by

$$[x_1 + \lambda(x_3 + x_4), x_2 + \lambda(x_3 + x_4), x_3, x_4].$$

We then find that the group of the points in the invariant plane, $x_4 = 0$, is of order $(p^k + 1)p^k(p^k - 1) \cdot p^{2k}$. There is in addition a group of elations of order p^k leaving fixed all the points in $x_4 = 0$. For in the group generated by (E_1) and (E_3) are contained all the transformations

$$[x_1 + \lambda(x_3 + x_4), x_2 - \lambda(x_3 - x_4), x_3, x_4].$$

The group conjugate with (E_3) under the reflection commutative with (E_1) and (E_2) is

$$[x_1 - \lambda(x_3 - x_4), x_2 + \lambda(x_3 - x_4), x_3, x_4].$$

As products of these we obtain a group of elations of order p^k having $(1\,0\,0\,0)$ and $x_4 = 0$ for center and axial plane. Hence the group generated by (E_1), (E_2), and (E_3) is of order $p^{4k}(p^{2k} - 1)$.

If now $(1\,0\,0\,0)$ and $x_4 = 0$ do not remain invariant there must be a group of elations (E_4) with center not in $x_4 = 0$. Its center must lie in at least one of each set of $p^k + 1$ planes similar to those considered above. Hence it must lie in one of the lines through $(1\,0\,0\,0)$ in which $p^k + 1$ of these planes meet. There are p^{2k} of these lines and they are all conjugate under the group generated by (E_1), (E_2), and (E_3). Hence we may take

a center on $x_2 = 0$, $x_3 = 0$, and since that group is invariant under any elation with center $(1\,0\,0\,0)$ and axial plane $x_4 = 0$, we may suppose it to be $(0\,0\,0\,1)$. The group (E_4) must then be given by

$$[x_1, x_2, x_3, \lambda x_1 + x_4].$$

Under the group now generated there are $p^{3k} + p^{2k} + p^k + 1$ conjugate groups of elations, their centers being the points with coördinates in the $GF(p^k)$. The subgroup which leaves $(1\,0\,0\,0)$ and $x_4 = 0$ invariant must be of order at least $1/2\,(p^k - 1) \cdot p^{4k}(p^{2k} - 1)$. For the group generated by (E_4) and the additive group having $(1\,0\,0\,0)$ and $x_4 = 0$ for center and axial plane must contain the transformation $[\epsilon x_1, x_2, x_3, \epsilon^{-1} x_4]$, where ϵ is a primitive root in the $GF(p^k)$. In $x_4 = 0$ this is an homology of period $p^k - 1$, whereas in the group generated by (E_1), (E_2), and (E_3) there are no homologies in that plane of higher period than 2.

The order of the whole group generated must then be at least

$$\tfrac{1}{2}(p^{4k} - 1)\,p^{4k}(p^{2k} - 1).$$

But this is exactly the order of $A_1(p^k)$ and all four groups of elations are contained by that group.

If $A_1(p^k)$ is a subgroup of $A_1(p^n)$, k must be a factor of n, and if $A_\nu(p^k)$ is a subgroup of $A_1(p^n)$, $2k$ must be a factor of n. The groups of either of these types form a single conjugate set under $A_\nu(p^n)$. If n/k is odd the groups of the type of $A_1(p^k)$ also form a single conjugate set under $A_1(p^n)$. If n/k is even, the groups of the type of $A_1(p^k)$ or $A_\nu(p^k)$ each form two conjugate sets under $A_1(p^n)$.

The subgroup of all transformations in $A_1(p^n)$ which leave invariant a point and a plane through it whose coördinates are in the $GF(p^n)$ is of order $1/2\,p^{4n}(p^{2n} - 1)(p^n - 1)$. The subgroup of all such transformations which leave invariant a line of the complex with coördinates in the $GF(p^n)$ is of the same order. The subgroup of all such transformations which leave invariant a congruence which is hyperbolic with respect to the $GF(p^n)$ is of order $p^{2n}(p^{2n} - 1)^2$. The groups of each of these three types form a single conjugate set under $A_1(p^n)$.

We still have to consider the case where $p^k = 3$ and the group generated by two elations is of order $2 \cdot 60$. Such a group is generated by

$$(E_1) : [x_1, x_2 + x_3, x_3, x_4]; \qquad (E_2) : [x_1, x_2, ix_2 + x_3, x_4],$$

where $i^2 = -1$. It may be proved just as in the general case that the centers of all elations which can exist must be in one of the axial planes of this group. We consider therefore a third elation, whose center and axial plane we may take to be $(1\,1\,0\,0)$ and $x_3 + x_4 = 0$. It must generate with (E_2) a group

of order $2\cdot 12$ or $2\cdot 60$. The reflection invariant under this group is

$$(R):[-x_1+2x_2, x_2, x_3+2x_4, -x_4].$$

The group generated by (E_1) and (E_2) contains the transformation

$$(S):[x_1, ix_2, -ix_3, x_4].$$

It is readily found that the group generated by the elations which are conjugate with (E_1) under the group generated by (R) and (S) contains a group of elations of order 3^2 having $(1\,0\,0\,0)$ and $x_4=0$ for center and axial plane. But this is contrary to our assumption that $p^k=3$.

4. Groups Containing Skew Elations but no Central Elations

We establish the following theorems:

Theorem 2. *Any subgroup of $A_1(p^n)$ which does not contain central elations cannot contain two skew elations with different axes if the axis of either is a fixed line of the other.*

If E_1 and E_2 are two skew elations with the same invariant complex and such that each leaves fixed the axis of the other, then it is easy to prove that $E_1^{-1}E_2^{-1}E_1E_2$ is a central elation. If E_1 leaves fixed the axis of E_2, but E_2 does not leave fixed the axis of E_1, then $E_2^{-1}E_1E_2$ and E_1 will have different axes but each will leave fixed the axis of the other. Hence a central elation will appear in this case also.

Theorem 3. *In any subgroup of $A_1(p^n)$ which does not contain central elations any two skew elations with different axes must leave invariant in common either all the lines of a regulus or the two axes of a congruence.*

Two skew elations with different axes must leave invariant the two axes of a congruence if their axes intersect, one of which is in the plane of their axes and the other of which passes through the point of intersection of their axes. If their axes do not intersect they must leave invariant at least one line in common, since their product can leave fixed no points which are not on a line left invariant by both. If they leave invariant only one line in common there must be in all cases among the products of their powers transformations with two fixed points on that line. Some power of such a transformation must be a skew elation having the invariant line for axis. But this is impossible, since its axis would be left invariant by the skew elations which generate the group. Hence the two generating skew elations must leave invariant either the two axes of a congruence or all the lines of a regulus.

Theorem 4. *Any subgroup of $A_1(p^n)$ which contains skew elations, but no central elations, and which leaves invariant the two axes of a congruence, must be $(1,1)$ or $(2,1)$ isomorphic with the groups of the points on the two axes, with the exception of three groups of order $2\cdot 4\cdot 3$, $2\cdot 4\cdot 12$, $2\cdot 4\cdot 24$, which exist for $p=3$.*

We notice that if there are any transformations which leave invariant all the points on one axis, the group generated by them must form an invariant subgroup. If there are any such transformations which are of period p on the second axis either they are central elations or some of their powers are central elations. The group on an axis cannot contain an invariant subgroup of order prime to p except for $p = 3$, in which case it must be of order 12 or 24. This readily leads to the exceptional groups noted in the theorem.

THEOREM 5. *Any subgroup of $A_1(p^n)$, which contains skew elations but no central elations, and which leaves invariant the two axes of a congruence, must contain subgroups of order $1/2\, p(p^2 - 1)$ which leave invariant all the lines of a regulus, provided that no point on either axis remains fixed.*

Any group of the sort described in the theorem must contain one or more subgroups which are of order $1/2p(p^2 - 1)$ on each axis of the congruence. In such a subgroup there will be skew elations whose product is of period 2 on each axis. But any such transformation in $A_1(p^n)$ may easily be shown to be a skew perspectivity leaving fixed all the points on each of two lines in the congruence. Since the product of the two skew elations can leave invariant no point which is not on a line left invariant by both, it follows that they must leave invariant in common all the lines of a regulus.

THEOREM 6. *Any subgroup of $A_1(p^n)$ containing an additive group of skew elations of higher order than 3, but no central elations, must leave invariant a point and plane, a congruence, or a quadric.*

An additive group of skew elations with a common axis will leave invariant either a pencil of lines (i. e., all the lines of the pencil) through each point of the axis, two pencils neither of which belongs to the complex, a single pencil belonging to the complex, or no pencils whatever. We denote the largest subgroup of the whole additive group which leaves invariant a pencil through each point of the axis by $H^{(R)}$ and suppose that its order is p^k.

If no point or line remains invariant under the group, $H^{(R)}$ must be contained either by a group leaving invariant all the lines of a regulus or by a group leaving invariant the two axes of a congruence. The latter must however contain subgroups of the former type and some of these will contain $H^{(R)}$. The largest group containing $H^{(R)}$ which leaves invariant all the lines of a regulus must be of order $1/2p^k(p^{2k} - 1)$, $p^k(p^{2k} - 1)$, or 60 $(p^k = 3)$. We denote one such group by $G^{(R)}$.

If there is another skew elation having the same axis as $H^{(R)}$, it must generate with $G^{(R)}$ a group having an invariant congruence. This congruence cannot be parabolic, since in that case there would be skew elations present having but one invariant line in common. The group generated must be of order $1/2p^m(p^{2m} - 1)$ or $p^m(p^{2m} - 1)$ on each axis of the congruence, m being a multiple of k. We denote this group by $G^{(C)}$ and its additive subgroup of order p^m, which is commutative with $H^{(R)}$, by $H^{(C)}$.

The group $G^{(C)}$ must contain subgroups of order $1/2(p^m - 1)$ or $p^m - 1$ on each axis of the congruence, which permute the operators of $H^{(C)}$ among themselves. Some of these will contain cyclic subgroups of order $1/2(p^k - 1)$ or $p^k - 1$ which belong to $G^{(R)}$. These are skew perspectivities which leave invariant all the points on each of two axes of elations in $G^{(R)}$. A group of order $1/2(p^m - 1)$ or $p^m - 1$ which leaves two such axes invariant must be of period on each of them greater than or equal to $p^k + 1$.* The period will clearly be greater than this unless $m = 2k$, in which case it may conceivably be $1/2(p^k + 1)$. In this case however $G^{(R)}$ would contain a skew perspectivity of order $p^k - 1$ and hence would be of order $p^k(p^{2k} - 1)$. It therefore contains also skew perspectivities of order $p^k + 1$. Under $G^{(C)}$ one of these would be commutative with the one of order $p^k - 1$. Hence the period on the two axes of elations could not in this case be less than $p^k + 1$.

If now there are other skew elations which have the same axis as $H^{(C)}$, there must be other groups of the same sort as $G^{(C)}$ containing $G^{(R)}$. There must then be more than one cyclic group leaving fixed two particular axes of elations in $G^{(R)}$ and the period of neither on these axes can be less than $p^k + 1$. The group generated by these cyclic groups must in all cases contain transformations which are of period p on each axis. Either such a transformation is a skew elation or else a power of it is a skew elation. But this is impossible since it would leave fixed the two axes of elations in $G^{(R)}$.

Hence in all cases the whole additive group of skew elations having a common axis must leave invariant either a pencil through each point of the axis or else two pencils. We will show that in the latter case a congruence (the invariant congruence of $G^{(C)}$) must remain invariant. Suppose this is not the case. Then there must be an elation with axis not in the congruence. It must generate with $H^{(R)}$ a group having either an invariant congruence or else an invariant regulus. In the former case there will be in the group generated subgroups of the latter type and not more than one of these can be in $G^{(C)}$. Any one of these must be of order $1/2p^k(p^{2k} - 1)$, and one of them must generate with $H^{(C)}$ a group of order $1/2p^m(p^{2m} - 1)$. The axes of the invariant congruence of this group will meet the axis of $H^{(C)}$ in the same points as the axes of the invariant congruence of $G^{(C)}$, i. e., the two points through which pass the two invariant pencils.

Hence $H^{(C)}$ will be left invariant by more than one metacyclic group, for the whole metacyclic group in a $G^{(C)}$ which leaves $H^{(C)}$ invariant can leave invariant but one congruence, as may be readily shown. The group generated by two transformations which permute the operators of $H^{(C)}$ in the same way, but which have different invariant congruences, will contain skew elations

* $G^{(R)}$ cannot now be of order 60, since $G^{(C)}$ would in that case contain a subgroup of order 360 permuting the same ten axes of elations and hence leaving the same regulus invariant.

with the same axis as $H^{(C)}$, which permute the lines of the two pencils. But this is impossible, as shown above.

We suppose finally that the entire additive group of skew elations is identical with $H^{(R)}$, i. e., leaves invariant a pencil of lines through each point of the axis. We will show that if $p^k > 3$ a regulus (the invariant regulus of $G^{(R)}$) must remain invariant. For if there be a skew elation not in $G^{(R)}$ it must generate with $H^{(R)}$ either a $G^{(C)}$ or another $G^{(R)}$. In the latter case the two $G^{(R)}$ must generate a $G^{(C)}$. If $p^k > 3$ every $G^{(C)}$ contains an additive group of order p^m, where m is a multiple of k. If $p^k = 3$ there are exceptional $G^{(C)}$ of order 60, $2\cdot 60$, $2\cdot 4\cdot 3$, $2\cdot 4\cdot 12$, $2\cdot 4\cdot 24$. That of order $2\cdot 4\cdot 3$ may be generated by two skew elations whose axes intersect and contains no $G^{(R)}$.

It is perhaps worth while here to discuss briefly the groups which have an invariant congruence or an invariant regulus. If the congruence is parabolic, or if it is hyperbolic with respect to the $GF(p^n)$, such groups are subgroups of subgroups which contain central elations. If the congruence is elliptic with respect to the $GF(p^n)$, there are no central elations in $A_1(p^n)$ which leave it invariant. The order of the largest subgroup of $A_1(p^n)$ which leaves one of these congruences invariant is $p^{2n}(p^{4n}-1)$. The totality of all such subgroups is a single conjugate set under $A_1(p^n)$.

There are two types of reguli, all the lines of which belong to the complex. In the one case there are two real lines of the conjugate regulus which belong to the complex and in the other these two lines are conjugate imaginary with respect to the $GF(p^n)$. The largest subgroup of $A_1(p^n)$ which leaves a regulus of the former type invariant is $(p^n+1)p^n(p^n-1)^2$. In the other case the largest subgroup is of order $(p^n+1)^2 p^n(p^n-1)$. The groups of each sort form a single conjugate set under $A_1(p^n)$. Neither of them is a maximal subgroup for $p^n = 3$. In that case the former leaves invariant both a hyperbolic and an elliptic congruence and the latter is a subgroup of a group of order 960 to be discussed later.

THEOREM 7. *If $p = 3$ every subgroup of $A_1(p^n)$ which contains skew elations but no central elations must leave invariant a point and plane, a congruence, or a regulus with the exception of two groups of order* $16\cdot 60$ *and* $16\cdot 120$. *The latter is a subgroup of $A_1(3^n)$ only if n is even.*

We first consider those subgroups which contain $G^{(C)}$ of order $2\cdot 4\cdot 12$ or $2\cdot 4\cdot 24$. Each of these contains C_4 which are the identity on one axis of the congruence and are of period 2 on the other. We will show that any reflection which is conjugate with the invariant reflection of this subgroup must be commutative with it, i. e., must interchange its axes.

Suppose first that there are two such reflections whose axes intersect. Their product will then be a skew elation. Consider a C_4 of the sort described above which is commutative with one of these reflections. It will either

transform the skew elation into another one having the same axis or will transform it into another one whose axis is left fixed by the first. In the first case there would be an additive group of skew elations present of higher order than 3. The second case is impossible by a previous theorem.

Suppose now that there are two such reflections whose axes do not intersect. If they do not interchange each other's axes their product cannot be of period less than 4. They must leave invariant in common all the lines of a regulus, since their axes are axes of congruences contained by the complex. A C_4 having one of them for its square must generate with the other a group which leaves two of the lines of this regulus invariant. If no additive group of higher order than 3 is present the group on each of these lines must be of order 24. But there must then be present reflections of the type considered whose products are skew elations.

Hence in any case two reflections which are squares of such C_4 must interchange each other's axes. The only groups which can contain $G^{(C)}$ of order $2\cdot4\cdot12$ or $2\cdot4\cdot24$ may then be shown* to be two groups of order $16\cdot60$ and $16\cdot120$ having an invariant G_{16}.

We suppose finally that the only $G^{(C)}$ which can be present are of order 60 or $2\cdot60$. We may then prove much as in Transactions I (under Theorem 13) that there must be commutative C_3. The proof that the R, A, and B of that discussion must leave invariant two lines in common is different, but may be made easily geometrically. But we have considered here the case where commutative C_3 are present.

5. Groups which Contain no Central Elations and no Skew Elations, but which Contain Skew Perspectivities of Higher Period than 2

The determination of the groups which contain no central elations or skew elations may be made much as in Transactions I, pp. 125–134. This is especially the case when the groups contain skew perspectivities of higher period than 2. Most of the general properties of the groups as stated there, p. 125, still hold if transformations of period p with a single fixed point are present. There are however the exceptions that if $p = 5$ two skew perspectivities of period 3 which generate a group of order 60, in which the operators of period 5 have each a single fixed point, leave invariant but one line in common. Also two reflections whose product is of period p can have but one fixed line in common. Much of the discussion is simplified on account of the limitation of the present discussion to those groups which have an invariant complex. Aside from this the differences are very slight. The results may be summarized as follows:

Theorem 8. *Any subgroup of $A_1(p^n)$ which does not contain central*

* Cf. Transactions I, p. 128.

elations or skew elations, but which contains skew perspectivities of higher period than 2, has either an invariant regulus or congruence or else is a group of one of the following orders: $16\cdot120$, $16\cdot60$, 360, 720, 2520, 5040. *Those of order* $16\cdot60$ *and* 360 *exist in every* $A_1(p^n)$, *that of order* $16\cdot120$ *only for* $p^n = 8h \pm 1$, *that of order* 720 *only for* $p^n = 12f \pm 1$, *that of order* 2520 *only for* $p = 7$, *and that of order* 5040 *only for* $p = 7$ *and* n *even.*

The groups occurring here which have invariant reguli or congruences are subgroups of those previously discussed. In the case of the groups of order $16\cdot120$ and $16\cdot60$, the axes of the five reflections contained by the invariant G_{16} which are axes of congruences must have coördinates in the $GF(p^n)$. The axes of the other ten are lines of the complex and must be real or conjugate imaginary with respect to the $GF(p^n)$ according as p^n has the form $4l + 1$ or $4l - 1$. These ten reflections are in the $G_{16\cdot120}$ the squares of skew perspectivities of period 4, which exist in $A_1(p^n)$ only if $p^n = 8h \pm 1$.

A $G_{16\cdot120}$ is generated by

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & d & 0 \\ b & 0 & 0 & d \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

where $a + d = 2$, $a^2 + 2b^2 + d^2 = 0$. These groups always exist in $A_r(p^n)$ and form a single conjugate set under that group, and (if they exist) two conjugate sets under $A_1(p^n)$. The subgroups of order $16\cdot60$ always exist in A_1, and if $p^n = 8h \pm 3$, they form a single conjugate set under A_1.

To determine when the other groups are subgroups of A_1 we distinguish two cases according as the axes of the skew perspectivities of period 3 are real or conjugate imaginary with respect to the $GF(p^n)$, i. e., according as $p^n = 3k + 1$ or $3k - 1$. The G_{360} may be generated by three such C_3 of which two are commutative and the third generates a G_{12} with one and a G_{60} with the other. If their axes are real we find that (if we consider conjugacy under A_r) they may be taken to be

$$(1000)(0100), \quad (0010)(0001);$$
$$(1000)(0010), \quad (0100)(0001);$$
$$(0111)(101-1), \quad (0-211)(-201-1).$$

A G_{720} containing this G_{360} must also contain the reflection whose axes are $(1\,i\,0\,0)(0\,0\,1\,i)$, $(1\,-i\,0\,0)(0\,0\,1\,-i)$, where $i^2 + 1 = 0$. These are axes of a congruence and hence the G_{720} are subgroups of A_1 only if they are real, i. e., if $p^n = 4l + 1$, and hence $p^n = 12f + 1$.

If the axes of the C_3 are conjugate imaginary with respect to the field, i. e.,

if $p^n = 3k - 1$, we may take the G_{360} to be generated by the three C_3, C_3', C_3'', C_3''' where one axis of C_3' is

$$(1\ 0\ 0\ \omega)(0\ 1\ \omega\ 0),$$

one axis of C_3'' is

$$(0\ 1\ \omega\ 0)(1\ 0\ 0\ \omega^2)$$

and one axis of C_3''' is

$$(1 - \lambda\alpha, -\alpha, -\alpha\omega^2, \omega^2 - \lambda\alpha\omega)(1, \alpha + \lambda, \alpha\omega^2 + \lambda\omega, \omega^2)$$

where $\omega^2 + \omega + 1 = 0$, $\alpha\alpha' = 1$, $\lambda\lambda' = -1$, α' and λ' being the conjugates of α and λ. The G_{720} must also contain the reflection whose axes are

$$(i\alpha, 1, \omega, i\alpha\omega)(1, i\alpha, i\alpha\omega^2, \omega^2)$$

and

$$(-i\alpha, 1, \omega, -i\alpha\omega)(1, -i\alpha, -i\alpha\omega^2, \omega^2).$$

These must be real and this is the case if i does not exist in the field, i. e., if $p^n = 4l - 1$ and hence also $p^n = 12f - 1$.

Hence A_1 has primitive subgroups of order 360 in all cases $(p > 3)$, whereas it has primitive subgroups of order 720 only if $p^n = 12f \pm 1$. These groups form each a single conjugate set under A_ν, whereas the G_{720} (if they exist) form two conjugate sets under A_1, and the G_{360} (if the G_{720} do not exist) form a single conjugate set under A_1.

The G_{2520} has an invariant complex only if $p = 7$ and the G_{5040} exists only in this case. The axes of the C_3 are then real and we may take the first choice of coördinates for a subgroup of order 360. The G_{2520} must also contain the C_3 whose axes are $(1\ 2\ 0\ 0)(0\ 0\ 3\ 1)$ and $(1\ 3\ 0\ 0)(0\ 0\ 2\ 1)$. The G_{5040} is generated by the G_{720} and this C_3. It appears in A_1 only if n is even. In this case A_1 contains two conjugate sets of G_{5040}, whereas if n is odd it contains a single conjugate set of G_{2520}.

6. Groups which Contain no Central Elations, no Skew Elations, and no Skew Perspectivities of Higher Period than 2

The results to be obtained in this section may be summarized in the following theorem.

Theorem 9. *The only subgroups of $A_1(p^n)$ which contain no central elations, no skew elations, and no skew perspectivities of higher period than 2, are subgroups either of groups previously considered or of a group of order $1/2p^n(p^{2n} - 1)$ having an invariant twisted cubic.*

The groups of this type whose orders are prime to p and which do not have an invariant congruence or regulus are of order $16\cdot5$, $16\cdot10$, $16\cdot20$, 60, 120. These are subgroups respectively of groups of order $16\cdot60$, $16\cdot60$, $16\cdot120$, 360, 720. It is only necessary to consider the groups which contain trans-

formations of period p. We may also in consequence of the discussion in Transactions I, p. 132, assume that these groups do not contain reflections whose axes are axes of congruences. We may also suppose that no transformation other than the identity can leave fixed the axes of more than one reflection. It may also easily be shown that two cyclic groups can have no transformations (other than the identity) in common unless both are contained by a larger cyclic group.

We consider an additive group of order p^m and suppose a transformation in this additive group to be

$$S : [x_1 + ax_2 + bx_3 + cx_4, x_2 + dx_3 + ex_4, x_3 + fx_4, x_4],$$

where $a \neq 0$, $d \neq 0$, $f \neq 0$. If $p_{14} + p_{23} = 0$ is invariant we must have $a + f = 0$, $b - e + df = 0$. If the additive group is invariant under a metacyclic group there must be present transformations such as

$$T : [\alpha x_1, \beta x_2, \beta^{-1} x_3, \alpha^{-1} x_4],$$

the period of which is a divisor of $p^m - 1$. The transformation $T^{-1}ST$ must be commutative with S. For otherwise if we denote $T^{-1}ST$ by S_1, then $S^{-1}S_1^{-1}SS_1$ would be either a central elation or a skew elation. This leads to the conditions $\alpha = \beta^3$, $b + e = 0$.

We inquire next under what conditions the cyclic group of order d_1 generated by T can be invariant under a group of order $4d_1$ which contains transformations making a cyclic permutation of period 4 on the vertices of its invariant tetrahedron. This leads to $\beta^{10} = 1$ and hence $d_1 = 5$.

If now we denote the order of the whole group by Ω, it is clear that there must be in the group $(p^m - 1)\Omega/d_1 p^m$ additive transformations. If $d_1 > 3$ any maximal cyclic group of order d_1 can be invariant only under groups of order d_1, $2d_1$, or $4d_1$, and in the last case we must have $d_1 = 5$. In the same way any other maximal cyclic group of order d_i can be invariant only under groups of order d_i, $2d_i$, or $4d_i$, and in the last case it may easily be shown that d_i can be divisible only by primes of the form $4l + 1$. There must then be $(d_i - 1)\Omega/f_i d_i$ transformations in each conjugate set of cyclic groups, where $f_i = 1, 2, 4$.

If we attempt to enumerate the transformations which the group must contain we are led to the diophantine equation

$$\Omega = 1 + (p^m - 1)\Omega/d_1 p^m + \sum_{i=1}^{r} (d_i - 1)\Omega/f_i d_i.$$

This equation is slightly different if the metacyclic group is of order $2p^m$ or $3p^m$, for transformations of period 2 or 3 which leave the additive group invariant may be contained by cyclic groups of higher order which do not

leave it invariant. Hence if the first denominator were $2p^m$ or $3p^m$ we could conclude only that d_1 is divisible by 2 or 3 respectively. The first would give an equation of essentially the same type as that considered in Transactions I under Theorem 14 and no solutions exist which correspond to groups. The second leads to either $r = 2, f_1 = f_2 = 2$ or $r = 2, f_1 = 2, f_2 = 4$.

Since p^m must have the form $6f + 1$, and since d_1 and d_2 can have no odd factor in common, it follows that in the first of these two cases $d_2 = 2$. If $d_1 = 6$, we obtain $\Omega = 3p^m$, which is impossible. Hence $d_1 \geqq 9$. This leads to $p^m < 12$ and hence $p^m = 7$. We then have $d_1 < 14$ and consequently $d_1 = 9, 12$. For $d_1 = 9$ there is no solution, whereas for $d_1 = 12$ we obtain $\Omega = 168$. This solution however can correspond to no group, since there would be seven cyclic groups of order 12 and hence the invariant reflection in any one of these groups would have to interchange the axes of each of the other six reflections of the same sort. This is easily seen to be impossible.

In the second case, since $f_2 = 4$, it follows that d_1 must be divisible by 12. Since Ω must be the least common multiple of $3p^m$, $2d_1$, and $4d_2$, we have $\Omega = 2p^m d_1 d_2$. This leads to

$$4d_1 d_2 + 6p^m d_2 + 3p^m d_1 = 6 + p^m d_1 d_2.$$

If $4d_1 d_2$ is the largest term on the left, it follows that $p^m < 12$ and hence $p^m = 7$. We then conclude that either $d_1 < 28$ or $d_2 < 14$, and hence that $d_1 = 12, 24$ or $d_2 = 5, 13$. None of these leads to a solution. The same result is obtained if $6p^m d_2$ or $3p^m d_1$ is supposed the largest term on the left.

We now take up the discussion of the general case. If no f_i has the value 4, the equation may be solved exactly as in a previous paper by the author.* The only solutions which need concern us here are:

$$r = 2, \quad f_1 = f_2 = 2, \quad d_1 = \tfrac{1}{2}(p^m - 1), \quad d_2 = \tfrac{1}{2}(p^m + 1),$$
$$\Omega = \tfrac{1}{2}p^m(p^{2m} - 1);$$
$$r = 2, \quad f_1 = f_2 = 2, \quad d_1 = p^m - 1, \quad d_2 = p^m + 1,$$
$$\Omega = p^m(p^{2m} - 1).$$

It will be shown later that there are groups which correspond to these two solutions.

We now suppose that at least one f_i has the value 4. If $f_1 = 4$, then (as shown above) $d_1 = 5$. Then one of the d's, say d_2, must be divisible by 4 and hence $f_2 = 1, 2$. It is evident from the equation that $f_2 = 2$ is the only possibility, since otherwise the sum of the coefficients of Ω on the right would exceed unity. It may then be seen from the equation that $r = 3$,

* These Transactions, vol. 12 (1911), p. 210.

$f_3 = 4$. Hence d_3 can be divisible only by primes of the form $4l + 1$, and since $d_1 = 5$, it follows that $d_3 \not< 13$. Since $p^m \not< 11$, we find that $d_2 = 4$. The order, Ω, of the group must be the least common multiple of the denominators, $5p^m$, 20, 8, $4d_3$, and hence must be $40d_3\, p^m$. This leads to

$$10p^m + 8d_3 = 1 + d_3\, p^m,$$

which may readily be shown to be impossible.

We now suppose that $f_1 = 1, 2$. If now any other d_i, say d_2, is even, we must have $f_2 = 1, 2$. This leads to $r = 2$, so that no f_i can be 4. Hence d_1 must be divisible by 4 and we then find $r = 3, f_1 = 2, f_2 = f_3 = 4$.

We observe that in this case the order of the group must be given by each of the two expressions, $\Omega = d_1\, p^m(1 + kp^m) = 2d_1\, d_2\, d_3\, p^m$. We may also obtain another expression for Ω by multiplying the number of reflections by the order of the group leaving a particular reflection invariant. There are d_1 reflections commutative with a particular reflection and $d_1 - 2$ other reflections commutative with each of these. The given reflection must be contained by two metacyclic groups of order $d_1\, p^m$, each of which contains $p^m - 1$ other reflections. It must also be contained by $d_1/2$ groups of order $4d_2$ and $d_1/2$ groups of order $4d_3$ which contain respectively $d_2 - 1$ and $d_3 - 1$ other reflections. Hence we must have

$$\Omega = 2d_1[1 + d_1(d_1 - 1) + 2(p^m - 1) + \tfrac{1}{2}d_1(d_2 - 1) + \tfrac{1}{2}d_1(d_3 - 1)].$$

Comparing this with one of the orders given above we find

$$2d_1(d_1 - 2) + (4p^m - 2) + d_1\, d_2 + d_1\, d_3 = p^m(1 + kp^m).$$

Also $2d_2\, d_3 = 1 + kp^m$. Since $d_1 < p^m$ and since we may suppose that $d_2 \not< 5$, $d_3 \not< 13$, it follows that

$$(4p^m - 2) + d_1\, d_2 + d_1\, d_3 < \tfrac{2\,2}{1\,3\,0}\, p^m(1 + kp^m).$$

Hence

$$2d_1(d_1 - 2) > \tfrac{1\,0\,8}{1\,3\,0}\, p^m(1 + kp^m).$$

This leads to $k = 1, 2$; $d_1 = p^m - 1$. We then find from the equation given above that $d_2 + d_3 - 4$ must be divisible by p^m. Hence we may suppose that $d_3 > p^m/2$. But this is impossible since $2d_2\, d_3 \not> 1 + 2p^m$. Hence no solution exists if any f_i has the value 4.

We proceed to construct the groups of order $1/2p^m(p^{2m} - 1)$ and $p^m(p^{2m} - 1)$. We may suppose that the former contains the transformations S and T written above, where $a + f = 0$, $b - e + df = 0$, $b + e = 0$, $\alpha = \beta^3$. We then obtain, by transforming S by the powers of T, the transformations

$$[x_1 + a\gamma x_2 + b\gamma^2 x_3 + c\gamma^3 x_4,\ x_2 + d\gamma x_3 + e\gamma^2 x_4,\ x_3 + f\gamma x_4,\ x_4],$$

where γ denotes any power of β^2. There are $1/2(p^m-1)$ such transformations and they must generate the group of order p^m. There must be among these transformations some whose product is also in the set unless $p=5$, $m=1$. This leads to the condition, $3c=-ab$. [This may also be proved in the special case by a consideration of the other additive group left invariant by T.]

If for the present we consider conjugacy only under $A_\nu(p^n)$, we may choose a and d to be arbitrary marks in the $GF(p^n)$ different from 0. We suppose therefore that $a=d=1$. The additive group then contains the transformations:

$$\left[x_1+tx_2+\frac{t^2}{2}x_3-\frac{t^3}{6}x_4,\ x_2+tx_3-\frac{t^2}{2}x_4,\ x_3-tx_4,\ x_4\right],$$

where t takes all values in the $GF(p^m)$.

The cyclic group generated by T must also leave another additive group invariant. Since the fixed points of the p^m+1 additive groups must be $(1\,0\,0\,0)$, $(0\,0\,0\,1)$, and the p^m-1 points $(-t^3/6,\ -t^2/2,\ -t,\ 1)$, we find this additive group to be

$$[x_1,\ 3tx_1+x_2,\ 6t^2x_1+4tx_2+x_3,\ -6t^3x_1-6t^2x_2-3tx_3+x_4].$$

The largest group of this type which belongs to $A_1(p^n)$ is that of order $1/2p^n(p^{2n}-1)$ obtained for $m=n$. They form a single conjugate set under $A_1(p^n)$ and may be said to correspond to the ordinary three-parameter continuous groups which have invariant twisted cubics.

7. Summary

We are now in a position to summarize our results. In the statement which follows we have, in a few places, denoted the order of $A_1(p^n)$ by $\Omega(p^n)$.

Theorem 10. *The maximal subgroups of $A_1(p^n)$ are as follows:*

(1) *groups of index $\Omega(p^n)/\Omega(p^k)$, each of which is conjugate under $A_1(p^n)$ with $A_1(p^k)$, where n/k is an odd prime;*

(2) *groups of index $\Omega(p^n)/2\Omega(p^k)$, each of which is conjugate under $A_\nu(p^n)$ with $A_\nu(p^k)$, where n is even and $n/k=2$;*

(3) *a single conjugate set of groups of index $p^{3n}+p^{2n}+p^n+1$, each of which has an invariant point and plane;*

(4) *a single conjugate set of groups of index $p^{3n}+p^{2n}+p^n+1$, each of which has an invariant parabolic congruence;*

(5) *a single conjugate set of groups of index $p^{2n}(p^{2n}+1)/2$, each of which has an invariant hyperbolic congruence;*

(6) *a single conjugate set of groups of index $p^{2n}(p^{2n}-1)/2$, each of which has an invariant elliptic congruence;*

(7) *a single conjugate set of groups of index* $p^{3n}(p^{2n}+1)(p^n+1)/2\,[p^n>3]$, *each of which has an invariant quadric such that all the lines of one regulus and two real lines of the other belong to the complex*;

(8) *a single conjugate set of groups of index* $p^{3n}(p^{2n}+1)(p^n-1)/2\,[p^n>3]$, *each of which has an invariant quadric such that all the lines of one regulus and two imaginary lines of the other belong to the complex*;

(9) *a single conjugate set of groups of index* $p^{3n}(p^{4n}-1)\,[p>3,\ p^n>7]$, *each of which has an invariant twisted cubic*;

(10) *two conjugate sets of groups of index* $\Omega(p^n)/16\cdot 120$ *if* $n=1, p=8h\pm 1$, *and a single conjugate set of groups of index* $\Omega(p^n)/16\cdot 60$ *if* $n=1,\ p=3$ *or* $8h\pm 3$.

(11) *two conjugate sets of groups of index* $\Omega(p^n)/720$ *if* $n=1, p=12f\pm 1$, *a single conjugate set of groups of index* $\Omega(p^n)/360$ *if* $n=1, p=5$ *or* $12f\pm 5$, $p\neq 7$, *and a single conjugate set of groups of index* $\Omega(p^n)/2520$ *if* $n=1$, $p=7$.

The G_{5040} is not a maximal subgroup of $A_1(7^n)$ for any n, but is a maximal subgroup of $A_\nu(7)$. Similarly for $p=12f\pm 5$, $p\neq 7$, the G_{720} is a maximal subgroup of $A_\nu(p)$, and if $p=3$ or $8h\pm 3$ the $G_{16\cdot 120}$ is a maximal subgroup of $A_\nu(p)$. For $p^n=5, 7$ the groups of index $p^{3n}(p^{4n}-1)$ are subgroups respectively of the G_{360} and the G_{2520}.

We observe that for $p^n=3$ the maximal subgroups are of index 40, 40, 45, 36, 27, which agrees with Dickson's result (loc. cit.). Also we find that the smallest index of any subgroup is $p^{3n}+p^{2n}+p^n+1$ except for $p^n=3$, in which case it is 27. Hence we have

THEOREM 11. *The smallest number of letters on which* $A_1(p^n)$ $[p$ *odd*$]$ *may be represented as a permutation group is* $p^{3n}+p^{2n}+p^n+1$ *except for* $p^n=3$, *in which case it is* 27.

This result is due to Jordan for $n=1$, $p=3$ and to Dickson* for $n=1$, $p>3$.

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* Loc. cit.

TRANSFORMATIONS OF CONJUGATE SYSTEMS WITH EQUAL POINT INVARIANTS*

BY

LUTHER PFAHLER EISENHART

In a recent memoir† we have established a transformation of a surface S into a surface S_1 such that the lines joining corresponding points form a congruence whose developables meet S and S_1 in conjugate systems with equal tangential invariants; we have called it a *transformation* Ω. In the present paper we consider a transformation of a surface S into a surface S_1 such that the developables of the congruence of joins of corresponding points meet S and S_1 in conjugate systems with equal point invariants. In both cases the fundamental operation is the transformation of Moutard‡ of an equation of the Laplace type with equal invariants, just as this transformation lies at the base of the various types of transformations in which the two surfaces are the focal sheets of a W-congruence. We identify the transformation under discussion with one which is involved in part in a theorem established from a different point of view by Koenigs, who at the time was considering an entirely different problem.§ We make a detailed study of this transformation,‖ because it seems to be the foundation of certain important transformations which have been established in other ways and it points the way to other important transformations. The transformations K admit a theorem of permutability just as the other two general types of transformations referred to above. So far as we know every class of transformations of surfaces of a particular kind into surfaces of the same kind is reducible to one of these three types. Consequently we not only know the basis for the existence of a theorem of permutability in the known particular transformations, but in subsequent investigations we shall have a more immediate method of establishing such a theorem for transformations under consideration. An example of this is given at the close of the memoir.

* Presented to the Society, April 25, 1914.

† *Conjugate systems with equal tangential invariants and the transformation of Moutard*, Rendiconti del circolo matematico di Palermo, vol. 38.

‡ Journal de l'école polytechnique, cahier 45 (1878), p. 1; cf. also, Darboux, *Leçons*, vol. 2, p. 145.

§ Comptes Rendus, vol. 113 (1891), p. 1022.

‖ Referred to in the following pages as a transformation K.

We show that transformations K are commutative with certain *radial transformations*, under which the points of a surface are transformed along lines which are concurrent. These transformations enable us to deduce the general theorem of permutability for transformations K from a special case in which the four surfaces consist of two pairs of associate surfaces, that is, two surfaces corresponding with parallelism of tangent planes and such that to asymptotic lines on either correspond a conjugate system on the other.

Although the treatment of §§ 1–5 is stated in terms of ordinary space, the results are equally true in n-space, if we substitute for a surface referred to a conjugate system a *reseau*, as defined by Guichard* and use the term *congruence* in the restricted sense given to it by this author.

The remainder of the memoir deals with ordinary space. It is shown that each pair of surfaces in the relation of a transformation K gives rise to four surfaces Σ, Σ_1, Σ_2, Σ', such that the pairs of surfaces $\Sigma\Sigma_1$, $\Sigma\Sigma_2$, $\Sigma_1\Sigma'$, $\Sigma_2\Sigma'$ are the focal surfaces of W-congruences, and conversely, when four surfaces are so related there exist four pairs of surfaces each pair of which are in the relation of a transformation K.

In order to apply the transformations K to particular types of surfaces it is desirable to put the equations in the form developed in § 9. The memoir contains one application, namely the solution of the problem of determining when S and S_1 envelop a two-parameter family of spheres. It is shown that in this case S and S_1 are isothermic surfaces in the relation of the transformation discovered by Darboux† and studied at length by Bianchi.‡ The theorem of permutability of these transformations is an immediate consequence of our general theorem.

The closing section of the memoir contains the solution of the problem under what conditions does the congruence of joins of corresponding points on two surfaces in the relation of a transformation K possess other pairs of points which describe surfaces similarly related. It is shown that, if a congruence possesses more than one pair of such points, it possesses an infinity of pairs.

In a subsequent memoir we shall consider the case where the joins of corresponding points on two surfaces S and S_1 form a normal congruence and we shall show that the surfaces orthogonal to such a congruence admit a transformation of the Ribaucour type.

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*Annales de l'école normale supérieure, sér. 3, vol. 14 (1897), pp. 467–516.

†Annales de l'école normale supérieure, sér. 3, vol. 16 (1899), pp. 491–508.

‡Annali di Matematica, ser. 3, vol. 11 (1905), pp. 93–158.

1. Fundamental transformation

By definition if x, y, z are the cartesian coördinates of a surface S referred to a conjugate system with equal point invariants, these functions are solutions of an equation of the form

$$\frac{\partial^2 \theta}{\partial u\,\partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v}\frac{\partial \theta}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u}\frac{\partial \theta}{\partial v} = 0, \tag{1}$$

where ρ is in general a function of u and v.

In accordance with the transformation of Moutard*, if θ is any solution of equation (1), the function ξ defined by the equations

$$\frac{\partial \xi}{\partial u} = -\rho\left(x\frac{\partial \theta}{\partial u} - \theta\frac{\partial x}{\partial u}\right), \qquad \frac{\partial \xi}{\partial v} = \rho\left(x\frac{\partial \theta}{\partial v} - \theta\frac{\partial x}{\partial v}\right) \tag{2}$$

is the solution of another equation with equal invariants. In fact, if we differentiate the first of (2) with respect to v, we find readily that ξ is a solution of the equation

$$\frac{\partial^2 \theta'}{\partial u\,\partial v} - \frac{\partial \log \sqrt{\rho}\,\theta}{\partial v}\frac{\partial \theta'}{\partial u} - \frac{\partial \log \sqrt{\rho}\,\theta}{\partial u}\frac{\partial \theta'}{\partial v} = 0. \tag{3}$$

If in (2) we replace ξ and x by the quantities η and y respectively, and also by ζ and z, the functions η and ζ so defined satisfy equation (3).

When the functions ξ, η, ζ are replaced by $\lambda x_1, \lambda y_1, \lambda z_1$, the necessary and sufficient condition that the surface S_1, whose coördinates are x_1, y_1, z_1, is referred to a conjugate system is that λ be a solution of equation (3). We assume that λ is such a function and write equations (2) in the form

$$\frac{\partial}{\partial u}(\lambda x_1) = -\rho\left(x\frac{\partial \theta}{\partial u} - \theta\frac{\partial x}{\partial u}\right), \qquad \frac{\partial}{\partial v}(\lambda x_1) = \rho\left(x\frac{\partial \theta}{\partial v} - \theta\frac{\partial x}{\partial v}\right). \tag{4}$$

It is readily shown that x_1, y_1, z_1 are solutions of

$$\frac{\partial^2 \phi}{\partial u\,\partial v} + \frac{\partial}{\partial v}\log\frac{\lambda}{\sqrt{\rho}\,\theta}\cdot\frac{\partial \phi}{\partial u} + \frac{\partial}{\partial u}\log\frac{\lambda}{\sqrt{\rho}\,\theta}\cdot\frac{\partial \phi}{\partial v} = 0. \tag{5}$$

Hence the parametric conjugate system on S_1 has equal point invariants.

We denote by M and M_1 corresponding points on S and S_1 and by x_0, y_0, z_0 the coördinates of a point M_0 on the line joining M and M_1. Evidently

$$x_0 = x + t(x_1 - x), \qquad y_0 = y + t(y_1 - y), \qquad z_0 = z + t(z_1 - z), \tag{6}$$

where t is a function of u and v.

By means of equations (4) the first derivatives of x_0 assume the form

* Loc. cit.

$$\frac{\partial x_0}{\partial u} = \left[t\left(\frac{\rho\theta}{\lambda} - 1\right) + 1\right]\frac{\partial x}{\partial u} + (x_1 - x)\left(\frac{\partial t}{\partial u} - t\frac{\partial \log \lambda}{\partial u}\right) - \frac{xt}{\lambda}\left(\frac{\partial \lambda}{\partial u} + \rho\frac{\partial \theta}{\partial u}\right),$$

(7)

$$\frac{\partial x_0}{\partial v} = \left[1 - t\left(1 + \frac{\rho\theta}{\lambda}\right)\right]\frac{\partial x}{\partial v} + (x_1 - x)\left(\frac{\partial t}{\partial v} - t\frac{\partial \log \lambda}{\partial v}\right) - \frac{xt}{\lambda}\left(\frac{\partial \lambda}{\partial v} - \rho\frac{\partial \theta}{\partial v}\right).$$

It is our purpose to show that λ can be so chosen that the developables of the congruence (G) of lines joining corresponding points M and M_1 meet the surfaces S and S_1 in the parametric conjugate systems with equal point invariants.

In the first place we observe that 1 is a solution of equation (1) and consequently it follows from (2), on replacing x by 1, that the equations

(8) $$\partial\lambda/\partial u = -\rho\,\partial\theta/\partial u, \qquad \partial\lambda/\partial v = \rho\,\partial\theta/\partial v$$

are consistent and that the function λ so defined is a solution of equation (3).

If we substitute this value of λ in equations (7) and give to t either of the values

(9) $$t_1 = \lambda/(\lambda - \rho\theta), \qquad t_2 = \lambda/(\lambda + \rho\theta),$$

we obtain respectively

$$\frac{\partial x_0/\partial u}{x_1 - x} = \frac{\partial y_0/\partial u}{y_1 - y} = \frac{\partial z_0/\partial u}{z_1 - z}, \qquad \frac{\partial x_0/\partial v}{x_1 - x} = \frac{\partial y_0/\partial v}{y_1 - y} = \frac{\partial z_0/\partial v}{z_1 - z}.$$

Hence the parametric ruled surfaces of the congruence (G) are developable.

Furthermore, if F_1 and F_2 denote the focal points whose coördinates are given by (6) when t is replaced by t_1 and t_2 from (9), it is readily shown that M and M_1 are harmonic to F_1 and F_2.

For the function λ given by equations (8) the equations (4) assume the form

(10) $$\frac{\partial x_1}{\partial u} = \frac{\rho}{\lambda}\left[(x_1 - x)\frac{\partial \theta}{\partial u} + \theta\frac{\partial x}{\partial u}\right], \qquad \frac{\partial x_1}{\partial v} = -\frac{\rho}{\lambda}\left[(x_1 - x)\frac{\partial \theta}{\partial v} + \theta\frac{\partial x}{\partial v}\right].$$

The foregoing results are stated in

THEOREM 1. *When a surface S referred to a conjugate system with equal point invariants is known, and also any solution θ of the point equation of S, the coördinates of a surface S_1 can be found by quadratures* (10), *and this surface S_1 is such that the developables of the congruence of lines joining corresponding points on S and S_1 meet these surfaces in conjugate systems with equal point invariants. Moreover, the focal points of the congruence are harmonic to the corresponding points on S and S_1.*

From (10) it follows that the necessary and sufficient condition that the tangents to the parametric curves on S and S_1 at corresponding points be parallel is that θ be constant. From equations (8) it follows that λ also must be constant. If we take $\theta = -\lambda = 1$, equations (10) become

$$\frac{\partial x_1}{\partial u} = -\rho \frac{\partial x}{\partial u}, \qquad \frac{\partial x_1}{\partial v} = \rho \frac{\partial x}{\partial v}. \tag{11}$$

These equations show that in this case S and S_1 are associate surfaces.* Moreover, it is known that the corresponding conjugate systems on any two associate surfaces have equal point invariants and their equations are of the form (11). Hence the problem of associate surfaces is a special case of the general problem which we are considering.

2. The theorem of Koenigs. The transformations K

In this section we show that by means of equations (10) we obtain the most general surface S_1 such that the congruence of lines joining corresponding points of S and S_1 possess the properties indicated in Theorem 1. We begin by looking upon the congruence (G) as composed of the tangents to the curves $v = \text{const.}$ on a surface S_0, whose coördinates are x_0, y_0, z_0 and on which the parametric system is conjugate.

Lucien Levy† has shown that the cartesian coördinates x, y, z of any point M of a line of the congruence (G) which describes a surface S cut in a conjugate system by the developables of (G) are expressible in the form

$$x = x_0 - \frac{\sigma}{\dfrac{\partial \sigma}{\partial u}} \frac{\partial x_0}{\partial u}, \qquad y = y_0 - \frac{\sigma}{\dfrac{\partial \sigma}{\partial u}} \frac{\partial y_0}{\partial u}, \qquad z = z_0 - \frac{\sigma}{\dfrac{\partial \sigma}{\partial u}} \frac{\partial z_0}{\partial u}, \tag{12}$$

where σ is a solution of the point equation of S_0, namely

$$\frac{\partial^2 \sigma}{\partial u \, \partial v} + a \frac{\partial \sigma}{\partial u} + b \frac{\partial \sigma}{\partial v} = 0. \tag{13}$$

Moreover, each solution of this equation leads to such a surface S.

We consider a second surface S_1 with coördinates x_1, y_1, z_1 given by

$$x_1 = x_0 - \frac{\sigma_1}{\dfrac{\partial \sigma_1}{\partial u}} \frac{\partial x_0}{\partial u}, \qquad y_1 = y_0 - \frac{\sigma_1}{\dfrac{\partial \sigma_1}{\partial u}} \frac{\partial y_0}{\partial u}, \qquad z_1 = z_0 - \frac{\sigma_1}{\dfrac{\partial \sigma_1}{\partial u}} \frac{\partial z_0}{\partial u}, \tag{14}$$

σ_1 being another solution of (13).

* E. p. 380. A reference of this kind is to the author's *Differential Geometry*, Ginn and Company, Boston (1909).

† *Sur quelques équations linéaires aux derivées partielles*, Journal de l'école polytechnique, cahier 56 (1886), p. 77.

For the sake of brevity we put

$$M = a + \frac{1}{\sigma}\frac{\partial \sigma}{\partial v}, \qquad N = b + \frac{1}{\sigma}\frac{\partial \sigma}{\partial u},$$

$$h = \frac{\partial a}{\partial u} + ab, \qquad k = \frac{\partial b}{\partial v} + ab. \tag{14'}$$

Thus h and k are the invariants of equation (13). It follows from (13) and (14') that

$$\frac{\partial M}{\partial u} = h - MN, \qquad \frac{\partial N}{\partial v} = k - MN. \tag{15}$$

In accordance with the theory of the Laplace transformation* of equation (13) the coördinates of the other focal point of the congruence (G) are of the form

$$x_0' = x_0 + \frac{1}{b}\frac{\partial x_0}{\partial u}.$$

The condition that the focal points be harmonic with respect to the corresponding points on S and S_1, whose coördinates are given by (12) and (14) is

$$N_1 + N = 0, \tag{16}$$

where N_1 is given by (14') when σ is replaced by σ_1. If we express the condition that N_1 satisfies an equation analogous to the second of (15), we have

$$M_1 = M - 2k/N, \tag{17}$$

where M_1 is given by (14') when σ is replaced by σ_1. If we require that M_1 satisfies an equation analogous to the first of (15) we get

$$\frac{\partial N}{\partial u} = N\frac{\partial \log k}{\partial u} + \frac{N^2}{k}(MN - k). \tag{18}$$

Whenever equations (13) and (18) admit a common solution σ, this function and σ_1, which follows by quadratures from (16) and (17), determine two surfaces S and S_1 in the relation of a transformation K.

Later (§ 11) we shall consider the conditions under which equations (13) and (18) possess a common solution. Now we are interested in showing that when these conditions are satisfied the coördinates of the two surfaces are in the relations expressed by equations (10).

From (12) we obtain by differentiation

$$\frac{\partial x}{\partial u} = \sigma\left(\frac{\partial \sigma}{\partial u}\right)^{-2}\left(\frac{\partial^2 \sigma}{\partial u^2}\frac{\partial x_0}{\partial u} - \frac{\partial \sigma}{\partial u}\frac{\partial x_0}{\partial v}\right),$$

$$\frac{\partial x}{\partial v} = \sigma\left(\frac{\partial \sigma}{\partial u}\right)^{-2} N\left(-\frac{\partial \sigma}{\partial v}\frac{\partial x_0}{\partial u} + \frac{\partial \sigma}{\partial u}\frac{\partial x_0}{\partial v}\right). \tag{19}$$

* E. p. 404.

The expression for $\partial^2 x/\partial u\,\partial v$ obtained from either of these equations can be put in the form

$$\frac{\partial^2 x}{\partial u\,\partial v} = \left(\frac{\partial \sigma}{\partial u}\right)^{-1}\left(N\frac{\partial \sigma}{\partial v}\frac{\partial x}{\partial u} + \frac{\frac{\partial \sigma}{\partial u}\left(\frac{\partial b}{\partial u} - b^2\right) - b\frac{\partial^2 \sigma}{\partial u^2}}{N}\frac{\partial x}{\partial v}\right). \tag{20}$$

A necessary and sufficient condition that this equation be of the form (1) is that there exist a function ρ defined by

$$\begin{aligned} \frac{\partial \log \sqrt{\rho}}{\partial u} &= -\left(\frac{\partial \sigma}{\partial u}\right)^{-1} N\frac{\partial \sigma}{\partial v}, \\ \frac{\partial \log \sqrt{\rho}}{\partial v} &= -\frac{\frac{\partial b}{\partial u} - b^2 - b\left(\frac{\partial \sigma}{\partial u}\right)^{-1}\frac{\partial^2 \sigma}{\partial u^2}}{N}. \end{aligned} \tag{21}$$

It is readily shown that these equations are consistent if equation (18) is satisfied. Owing to the fact that the relation between S and S_1 is reciprocal, it follows that the parametric system on S_1 also is conjugate with equal point invariants. Hence we have the theorem of Koenigs:*

THEOREM 2. *If two surfaces S and S_1 are so related that the focal points of the congruence of lines joining corresponding points M and M_1 on these surfaces are harmonic with respect to these points, and also if the developables of the congruence meet S and S_1 in conjugate systems, the latter have equal point invariants.*

Now we show that coördinates the of S and S_1 are in the relation of equations (10).

Since σ_1 determined by quadratures from (16) and (17) is a solution of equation (13), it is evident from (12) that the function θ given by

$$\theta = \sigma_1 - \frac{\sigma}{\frac{\partial \sigma}{\partial u}}\frac{\partial \sigma_1}{\partial u} = 2\sigma_1\left(1 + b\frac{\sigma}{\frac{\partial \sigma}{\partial u}}\right) \tag{22}$$

is a solution of equation (20).

The first derivatives of θ are given by equations obtained from equations (19) on replacing x_0 by σ_1; and from these same equations on substituting σ_1 for σ we get the first derivatives of x_1. When these values are substituted in equations (10), they are satisfied, provided that

$$\lambda = \rho\left(\sigma_1 - \frac{\sigma}{\frac{\partial \sigma}{\partial u}}\frac{\partial \sigma_1}{\partial u}\right)\frac{\sigma}{\frac{\partial \sigma}{\partial u}}\frac{\partial \sigma_1}{\partial u}\Big/\sigma_1 = \rho\frac{\theta}{\sigma_1}(\sigma_1 - \theta). \tag{23}$$

When this expression for λ is substituted in equations (8), we are led to equations (21). Hence we have

* Loc. cit.

THEOREM 3. *The transformation* (10) *of a surface* S *into a surface* S_1 *is the most general case of pairs of surfaces in the relation of Koenigs.*

In view of this result we shall hereafter say that S_1 given by (10) is obtained from S by a *transformation* K.

3. THEOREM OF PERMUTABILITY

Suppose that we have two solutions θ_1 and θ_2 of equation (1) and consider the two surfaces S_1 and S_2 which arise from S by transformations K_1 and K_2 determined by θ_1 and θ_2 respectively. The coördinates x_1, y_1, z_1 and x_2, y_2, z_2 are given by equations of the form (10), namely

$$\begin{aligned} \frac{\partial x_i}{\partial u} &= \frac{\rho}{\lambda_i}\left[(x_i - x)\frac{\partial \theta_i}{\partial u} + \theta_i \frac{\partial x}{\partial u}\right], \\ \frac{\partial x_i}{\partial v} &= -\frac{\rho}{\lambda_i}\left[(x_i - x)\frac{\partial \theta_i}{\partial v} + \theta_i \frac{\partial x}{\partial v}\right] \end{aligned} \qquad (i = 1, 2), \tag{24}$$

where, in accordance with (8),

$$\frac{\partial \lambda_i}{\partial u} = -\rho \frac{\partial \theta_i}{\partial u}, \qquad \frac{\partial \lambda_i}{\partial v} = \rho \frac{\partial \theta_i}{\partial v} \qquad (i = 1, 2). \tag{25}$$

From (5) it follows that the functions x_i, y_i, z_i, defined by (24) and similar equations in y_i and z_i, satisfy the corresponding equation

$$\frac{\partial^2 \phi_i}{\partial u\, \partial v} + \frac{\partial}{\partial v}\log\frac{\lambda_i}{\sqrt{\rho}\,\theta_i}\frac{\partial \phi_i}{\partial u} + \frac{\partial}{\partial u}\log\frac{\lambda_i}{\sqrt{\rho}\,\theta_i}\frac{\partial \phi_i}{\partial v} = 0 \qquad (i = 1, 2). \tag{26}$$

It is evident from (4) that these equations are satisfied also by the functions θ_1' and θ_2' respectively, defined by

$$\begin{aligned} \frac{\partial}{\partial u}(\theta_i' \lambda_i) &= -\rho\left(\theta_i \frac{\partial \theta_j}{\partial u} - \theta_j \frac{\partial \theta_i}{\partial u}\right), \\ \frac{\partial}{\partial v}(\theta_i' \lambda_i) &= \rho\left(\theta_i \frac{\partial \theta_j}{\partial v} - \theta_j \frac{\partial \theta_i}{\partial v}\right) \end{aligned} \qquad \begin{pmatrix} i = 1, 2, \\ j = 1, 2; \end{pmatrix} \quad i \neq j). \tag{27}$$

As thus defined each of the functions θ_i' is determined only to within the additive function c_i/λ_i, where c_i is an arbitrary constant. Hereafter in using pairs of functions θ_1' and θ_2' simultaneously we understand that the constants c_i are so determined that

$$\theta_1' \lambda_1 + \theta_2' \lambda_2 = 0, \tag{28}$$

which evidently is possible because of the form of (27).

It is our purpose to show that, if we use the functions θ_1' and θ_2' to determine transformations K_1' and K_2' of S_1 and S_2 respectively, the resulting surfaces coincide, giving a surface S' with coördinates x', y', z'.

In the first place we observe that if we put

$$\rho_i = \lambda_i^2/\rho\theta_i^2, \tag{29}$$

equations (26) may be given a form similar to (1). Hence for the transformations K_1' and K_2' the equations analogous to (25) are

$$\frac{\partial\lambda_i'}{\partial u} = -\frac{\lambda_i^2}{\rho\theta_i^2}\frac{\partial\theta_i'}{\partial u}, \qquad \frac{\partial\lambda_i'}{\partial v} = \frac{\lambda_i^2}{\rho\theta_i^2}\frac{\partial\theta_i'}{\partial v} \qquad (i = 1, 2). \tag{30}$$

In order that the transforms of S_1 and S_2 may coincide, equations analogous to (24) necessitate the conditions

$$\begin{aligned} \frac{\partial x'}{\partial u} &= \frac{\rho_1}{\lambda_1'}\left[(x' - x_1)\frac{\partial\theta_1'}{\partial u} + \theta_1'\frac{\partial x_1}{\partial u}\right] = \frac{\rho_2}{\lambda_2'}\left[(x' - x_2)\frac{\partial\theta_2'}{\partial u} + \theta_2'\frac{\partial x_2}{\partial u}\right], \\ \frac{\partial x'}{\partial v} &= -\frac{\rho_1}{\lambda_1'}\left[(x' - x_1)\frac{\partial\theta_1'}{\partial v} + \theta_1'\frac{\partial x_1}{\partial v}\right] = -\frac{\rho_2}{\lambda_2'}\left[(x' - x_2)\frac{\partial\theta_2'}{\partial v} + \theta_2'\frac{\partial x_2'}{\partial v}\right]. \end{aligned} \tag{31}$$

If we substitute in the second equation of the first row the expressions for $\partial\theta_1'/\partial u$ and $\partial x_i/\partial u$ given by (24) and (27), we obtain

$$\left(\theta_2\frac{\partial\theta_1}{\partial u} - \theta_1\frac{\partial\theta_2}{\partial u}\right)P + \frac{\lambda_1\theta_1'}{\theta_2}\left(\frac{1}{\lambda_1'\theta_1} + \frac{1}{\lambda_2'\theta_2}\right)\left[(x' - x)\frac{\partial\theta_2}{\partial u} + \theta_2\frac{\partial x}{\partial u}\right] = 0, \tag{32}$$

where for the sake of brevity we have put

$$P = x'\left[\frac{\lambda_1}{\lambda_1'\theta_1^2\theta_2}(\theta_1' + \theta_2) + \frac{\lambda_2}{\lambda_2'\theta_2'}\right] - \left(\frac{x_1\lambda_1}{\lambda_1'\theta_1^2} + \frac{x_2\lambda_2}{\lambda_2'\theta_2^2}\right) - \frac{\lambda_1\theta_1' x}{\theta_1^2\theta_2\lambda_1'}. \tag{33}$$

In a similar manner we obtain from the second row of (31) the equation

$$\left(\theta_2\frac{\partial\theta_1}{\partial v} - \theta_1\frac{\partial\theta_2}{\partial v}\right)P + \frac{\lambda_1\theta_1'}{\theta_2}\left(\frac{1}{\lambda_1'\theta_1} + \frac{1}{\lambda_2'\theta_2}\right)\left[(x' - x)\frac{\partial\theta_2}{\partial v} + \theta_2\frac{\partial x}{\partial v}\right] = 0. \tag{34}$$

The equations of condition (32) and (34) are satisfied if $P = 0$ and if

$$\lambda_1'\theta_1 + \lambda_2'\theta_2 = 0. \tag{35}$$

We shall show that these conditions are satisfied and lead to the desired result.

If we differentiate equation (35) with respect to u and v separately and require that the functions λ_i' shall satisfy (30), we find that we must have

$$\lambda_1'\theta_1 = \lambda_1(\theta_1' + \theta_2) - \lambda_2\theta_1, \qquad \lambda_2'\theta_2 = \lambda_2(\theta_2' + \theta_1) - \lambda_1\theta_2. \tag{36}$$

In consequence of (28) these equations are consistent with (35) and furthermore these expressions for λ_i' satisfy equations (30).

Because of (36) the equation obtained by equating P to zero is reducible to

$$\lambda_1'\theta_1 x' = \theta_1'\lambda_1 x + \lambda_1\theta_2 x_1 - \lambda_2\theta_1 x_2, \tag{37}$$

which, in consequence of (28) and (35), is equivalent to

$$\lambda_2' \theta_2 x' = \theta_2' \lambda_2 x + \lambda_2 \theta_1 x_2 - \lambda_1 \theta_2 x_1. \tag{38}$$

With the aid of equations (36) it is shown that the value of x' given by (37) satisfies equations (31). Hence not only does S' exist satisfying the above requirements, but its coördinates can be found without quadrature after θ_1' and θ_2' have been obtained by the quadratures (27).

We shall say that four surfaces S, S_1, S_2, S' in the above relation form a *quatern*. From (37) it is evident that the corresponding points on the four surfaces lie in a plane.

From the form of (27) it is seen that θ_1' is determined to within the additive quantity c/λ_1, where c is an arbitrary constant. The first of equations (36) shows that the corresponding change in λ_1' is the addition of c/θ_1. Hence if θ_1' is a function giving a surface S' and if we denote by S_c' the surface determined by $\theta_1' + c/\lambda_1$, the coördinates x_c', y_c', z_c' of S_c' are expressible in the form

$$(\lambda_1' \theta_1 + c) x_c' = (\theta_1' \lambda_1 + c) x + \lambda_1 \theta_2 x_1 - \lambda_2 \theta_1 x_2,$$

which may be written also

$$(\lambda_1' \theta_1 + c) x_c' = cx + \lambda_1' \theta_1 x'. \tag{39}$$

Hence all the corresponding points on the family of surfaces S_c' lie on the line joining M and M', and each point divides the segment MM' in the ratio $c/\lambda_1' \theta_1$.

The foregoing results may be assembled into

THEOREM 4. *If S_1 and S_2 are two surfaces arising from S by transformations K, there can be found by a quadrature an infinity of surfaces S' each of which is in the relation of transformations K with both S_1 and S_2. The corresponding points on the suite of surfaces S' lie on a line which passes through the corresponding point on S and in the plane determined by the latter point and the corresponding points on S_1 and S_2.*

We consider in particular the case when S_1 is an associate surface of S. As shown in § 1 this arises when we take $\theta_1 = -\lambda_1 = 1$. Now equations (27) for $i = 1$ become

$$\frac{\partial \theta_1'}{\partial u} = \rho \frac{\partial \theta_2}{\partial u}, \qquad \frac{\partial \theta_1'}{\partial v} = -\rho \frac{\partial \theta_2}{\partial v}.$$

Comparing these with (25) and making use of (28) we see that

$$\theta_1' = -\lambda_2 + c, \qquad \theta_2' = -1 + c/\lambda_2.$$

Hence if we take $c = 0$ the surfaces S_2 and S' are associate. In this case

equation (37) can be written

$$\theta_2(x' - x_1) = \lambda_2(x_2 - x). \tag{40}$$

Therefore we have

THEOREM 5. *If S and S_1 are two associate surfaces, and if S_2 is any surface arising from S by a transformation K, there exists a surface S' associate to S_2 which is in the relation of a transformation K with S_1; moreover, the lines joining corresponding points of S_1 and S' and of S and S_2 are parallel.*

4. ENVELOPE OF THE PLANES OF A QUATERN

We have just seen that the four corresponding points M, M_1, M_2, M' of the surfaces of a quatern S, S_1, S_2, S' lie in a plane. Since this plane contains the lines MM_1 and MM_2 which generate congruences whose developables meet S in a conjugate system, it envelopes a surface upon which the curves corresponding to these developables form a conjugate system, as follows from the general theory of conjugate systems and congruences.* Moreover, if Π is the point of the envelope corresponding to the point M on S, the tangent at Π to one of these curves passes through the focal points F_1' and F_2' of the lines MM_1 and MM_2 respectively, and the tangent to the other curve passes through the other focal points F_1'' and F_2''. Since the set of points M, M_1, F_1', F_{11}'' are harmonic and likewise the points M, M_2, F_2', F_2'', it follows that the line joining M_1 and M_2 passes through Π. As the surfaces S and S' bear to S_1 a relation similar to that of S_1 and S_2 to S, it follows that the point Π lies also on the line MM'.

If X, Y, Z denote the coördinates of Π, in order that it be the intersection of the lines MM' and M_1M_2, there must exist functions σ and τ such that

$$X = x + \sigma(x' - x) = x_1 + \tau(x_2 - x_1), \tag{41}$$

and similar expressions for Y and Z. If we eliminate x' from equation (37) and from the equation formed by the last two members of (41), we obtain an equation of the form $ax + bx_1 + cx_2 = 0$. Since the same equation is satisfied by the y's and z's, the expressions a, b, and c must be zero. These equations are consistent in consequence of (36) and give the expressions

$$\sigma = \frac{\lambda_1'\theta_1}{\lambda_1'\theta_1 - \lambda_1\theta_1'}, \qquad \tau = -\frac{\lambda_2\theta_1}{\lambda_1\theta_2 - \lambda_2\theta_1}.$$

When these values are substituted in (41), we obtain

$$X = \frac{\lambda_1'\theta_1 x' - \theta_1'\lambda_1 x}{\lambda_1'\theta_1 - \theta_1'\lambda_1} = \frac{\lambda_1\theta_2 x_1 - \lambda_2\theta_1 x_2}{\lambda_1\theta_2 - \lambda_2\theta_1}, \tag{42}$$

which are consistent because of (37).

* Cf. Guichard, loc. cit., p. 491.

From (6) and (9) it follows that the coördinates of the focal points, namely x_i', y_i', z_i' and x_i'', y_i'', z_i'' for $i = 1, 2$, are given by expressions of the form

$$x_i' = \frac{\lambda_i x_i - \rho\theta_i x}{\lambda_i - \rho\theta_i}, \qquad x_i'' = \frac{\lambda_i x_i + \rho\theta_i x}{\lambda_i + \rho\theta_i} \qquad (i = 1, 2). \tag{43}$$

Hence the direction-parameters of the lines $F_1' F_2'$, $F_1'' F_2''$ are proportional to expressions of the form

$$\begin{aligned} &x_1 \lambda_1 (\lambda_2 - \rho\theta_2) - x_2 \lambda_2 (\lambda_1 - \rho\theta_1) + \rho x (\lambda_1 \theta_2 - \lambda_2 \theta_1), \\ &x_1 \lambda_1 (\lambda_2 + \rho\theta_2) - x_2 \lambda_2 (\lambda_1 + \rho\theta_1) - \rho x (\lambda_1 \theta_2 - \lambda_2 \theta_1), \end{aligned} \tag{44}$$

respectively. With the aid of these values it is readily verified that the point whose coördinates are given by (42) is the intersection of the lines $F_1' F_2'$ and $F_1'' F_2''$.

From the second expression (42) for X we obtain by differentiation

$$\begin{aligned} \frac{\partial X}{\partial u} &= [x_1 \lambda_1 (\lambda_2 + \rho\theta_2) - x_2 \lambda_2 (\lambda_1 + \rho\theta_1) + \rho x (\theta_1 \lambda_2 - \theta_2 \lambda_1)] K_1, \\ \frac{\partial X}{\partial v} &= [x_1 \lambda_1 (\lambda_2 - \rho\theta_2) - x_2 \lambda_2 (\lambda_1 - \rho\theta_1) - \rho x (\theta_1 \lambda_2 - \theta_2 \lambda_1)] K_2, \end{aligned} \tag{45}$$

where

$$K_1 = \frac{\theta_2 \dfrac{\partial \theta_1}{\partial u} - \theta_1 \dfrac{\partial \theta_2}{\partial u}}{(\theta_1 \lambda_2 - \theta_2 \lambda_1)^2}, \qquad K_2 = \frac{\theta_2 \dfrac{\partial \theta_1}{\partial v} - \theta_1 \dfrac{\partial \theta_2}{\partial v}}{(\theta_1 \lambda_2 - \theta_2 \lambda_1)^2}. \tag{46}$$

Comparing equations (44) and (45), we note that the curves $v = \text{const.}$ on the envelope are tangent to the corresponding lines $F_1'' F_2''$ and the curves $u = \text{const.}$ to the lines $F_1' F_2'$, as previously remarked from the general theory. As the congruences of lines $M' M_1$ and $M' M_2$ bear to the surface S' a relation similar to that of the lines MM_1 and MM_2 to S, the focal points on the lines $M' M_1$ and $M' M_2$ lie on the tangents to the curves $v = \text{const.}$ or $u = \text{const.}$ It follows that the two tangents at a point Π are harmonic with respect to the lines MM' and $M_1 M_2$ through the point.

In order to find the equation satisfied by the coördinates of Π, we differentiate the first of equations (45) with respect to v. After direct reductions we find that

$$\begin{aligned} &\frac{\partial^2 X}{\partial u\, \partial v} + \frac{K_2}{K_1}\left[\frac{\partial \log \sqrt{\rho}}{\partial u} + \frac{\lambda_2 \dfrac{\partial \theta_1}{\partial u} - \lambda_1 \dfrac{\partial \theta_2}{\partial u}}{\theta_1 \lambda_2 - \theta_2 \lambda_1}\right] \frac{\partial X}{\partial u} \\ &\qquad + \frac{K_1}{K_2}\left[\frac{\partial \log \sqrt{\rho}}{\partial v} + \frac{\lambda_2 \dfrac{\partial \theta_1}{\partial v} - \lambda_1 \dfrac{\partial \theta_2}{\partial v}}{\theta_1 \lambda_2 - \theta_2 \lambda_1}\right] \frac{\partial X}{\partial v} = 0. \end{aligned} \tag{47}$$

It is evident that ordinarily the invariants of this equation are not equal.

We inquire whether the point Π ever coincides with one of the suite defined by equation (39) for all values of u and v. If we equate this value of x_c' to the second expression (42) for X, we are led to the condition $\theta_1' \lambda_1 = -c$, which by (27) necessitates $\theta_2/\theta_1 = \text{const.}$ In this case S_1 and S_2 are homothetic.

Gathering together the above results we have

THEOREM 6. *If S, S_1, S_2, S' are four surfaces of a quatern, the plane Π of four corresponding points M, M_1, M_2, M' touches its envelope in the intersection Π of the lines MM' and $M_1 M_2$; the parametric lines on the envelope form a conjugate system (with invariants ordinarily unequal) whose tangents are harmonic to the lines MM' and $M_1 M_2$ and contain the focal points of the lines MM_1, MM_2, $M' M_1$, $M' M_2$.*

5. RELATIONS BETWEEN TRANSFORMATIONS K AND RADIAL TRANSFORMATIONS

If ω is a solution of equation (1) which is linearly independent of x, y, z, the surface $\bar{S}$ whose coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$ are given by

$$\bar{x} = x/\omega, \qquad \bar{y} = y/\omega, \qquad \bar{z} = z/\omega \tag{48}$$

is referred to a conjugate system with equal point invariants. In fact, it is easily shown that the coördinates of $\bar{S}$ satisfy the equation

$$\frac{\partial^2 \bar{\theta}}{\partial u\, \partial v} + \frac{\partial}{\partial v} \log \sqrt{\rho}\, \omega \frac{\partial \bar{\theta}}{\partial u} + \frac{\partial}{\partial u} \log \sqrt{\rho}\, \omega \frac{\partial \bar{\theta}}{\partial v} = 0. \tag{49}$$

We shall say that $\bar{S}$ is obtained from S by a *radial transformation.*

If θ is a function linearly independent of x, y, z, and ω, we shall find that the surfaces S_1 and $\bar{S}_1$ resulting from S and $\bar{S}$ by transformations K, determined by θ and $\bar{\theta}$ respectively, where

$$\bar{\theta} = \theta/\omega, \tag{50}$$

are in the relation of a radial transformation. In fact, we shall show that the coördinates x_1, y_1, z_1 and $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$ of S_1 and $\bar{S}_1$ respectively are in the relations

$$\bar{x}_1 = x_1/\omega_1, \qquad \bar{y}_1 = y_1/\omega_1, \qquad \bar{z}_1 = z_1/\omega_1, \tag{51}$$

where ω_1 is given by the quadratures

$$\frac{\partial \omega_1}{\partial u} = \frac{\rho}{\lambda}\left[(\omega_1 - \omega)\frac{\partial \theta}{\partial u} + \theta \frac{\partial \omega}{\partial u}\right], \qquad \frac{\partial \omega_1}{\partial v} = -\frac{\rho}{\lambda}\left[(\omega_1 - \omega)\frac{\partial \theta}{\partial v} + \theta \frac{\partial \omega}{\partial v}\right]. \tag{52}$$

As equations (52) are of the form of (10), it is evident that they are consistent and that ω_1 is a solution of the point equation of S_1, namely (5).

The equations of the transformation from $\bar{S}$ to $\bar{S}_1$, analogous to (10), are

$$\begin{aligned}\frac{\partial \bar{x}_1}{\partial u} &= \frac{\rho\omega^2}{\bar{\lambda}}\left[\left(\frac{x_1}{\omega_1}-\frac{x}{\omega}\right)\frac{\partial}{\partial u}\left(\frac{\theta}{\omega}\right)+\frac{\theta}{\omega}\frac{\partial}{\partial u}\left(\frac{x}{\omega}\right)\right],\\ \frac{\partial \bar{x}_1}{\partial v} &= -\frac{\rho\omega^2}{\bar{\lambda}}\left[\left(\frac{x_1}{\omega_1}-\frac{x}{\omega}\right)\frac{\partial}{\partial v}\left(\frac{\theta}{\omega}\right)+\frac{\theta}{\omega}\frac{\partial}{\partial v}\left(\frac{x}{\omega}\right)\right].\end{aligned} \tag{53}$$

When the value of $\bar{x}_1$ from (51) is substituted in (53) and in the reduction equations (52) are used, it is found that the necessary and sufficient condition that these equations be satisfied is that

$$\bar{\lambda} = \omega_1 \lambda. \tag{54}$$

With the aid of equations (52) it is readily shown that this value of $\bar{\lambda}$ satisfies the equations

$$\frac{\partial \bar{\lambda}}{\partial u} = -\rho\omega^2 \frac{\partial}{\partial u}\left(\frac{\theta}{\omega}\right), \qquad \frac{\partial \bar{\lambda}}{\partial v} = \rho\omega^2 \frac{\partial}{\partial v}\left(\frac{\theta}{\omega}\right), \tag{55}$$

which are analogous to equations (8). Hence we have

THEOREM 7. *If two surfaces S and $\bar{S}$ are in the relation of a radial transformation and a transformation K of S is known, it is possible to find by quadratures a transformation K of $\bar{S}$ such that the new surfaces S_1 and $\bar{S}_1$ are in the relation of a radial transformation.*

If in particular $\theta = \omega$, it follows from (55) that $\bar{\lambda}$ is constant. When we take $\bar{\lambda} = -1$, equation (54) becomes

$$\omega_1 = -1/\lambda, \tag{56}$$

which we have seen in § 3 is a solution of equation (5). With the aid of equations (8) it can be shown that this value of ω_1 satisfies equations (52). Now $\bar{\theta} = 1$, consequently $\bar{S}_1$ is associate to $\bar{S}$, as shown in § 1. From these results we have

THEOREM 8. *If a surface S is subjected to a radial transformation determined by a function θ, and if the associate of the resulting surface, determined by the parametric conjugate system, undergoes a radial transformation determined by $-\lambda$, where λ is given by the quadratures* (8), *the new surface S_1 and S are in the relation of the transformation K determined by θ.*

Suppose now that we have a quatern of surfaces S, S_1, S_2, S', expressed in terms of the same functions appearing in § 3. If θ_2 determines a radial transformation of S, a comparison of equations (52) and (27) reveals the fact that the function $\omega_1 = -\theta_1'$. On the other hand $\omega_2 = -1/\lambda_2$. Again, since the radial transformation function of S_1 differs only in sign from the function θ_1' of the transformation of S_1 into S', it follows that $\omega' = 1/\lambda_1'$.

Hence the four surfaces $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$, $\bar{S}'$, whose x-coördinates are respectively of the form

$$\bar{x} = x/\theta_2, \qquad \bar{x}_1 = -x_1/\theta_1', \qquad \bar{x}_2 = -\lambda_2 x_2, \qquad \bar{x}' = \lambda_1' x', \tag{57}$$

are radial transforms of the surfaces of the original quatern; and these surfaces themselves form a quatern of the special type of Theorem 5, the surfaces $\bar{S}$ and $\bar{S}_2$ being associate and likewise $\bar{S}_1$ and $\bar{S}'$. The transformation functions of this new quatern are

$$\begin{gathered}\bar{\theta}_1 = \theta_1/\theta_2, \qquad \bar{\lambda}_1 = -\lambda_1\theta_1'; \qquad \bar{\theta}_2 = -\bar{\lambda}_2 = 1; \qquad \bar{\theta}_1' = -\bar{\lambda}_1' = -1; \\ \bar{\theta}_2' = \lambda_1\theta_1', \qquad \bar{\lambda}_2' = -\theta_1/\theta_2.\end{gathered} \tag{58}$$

It is readily shown that these functions and the coördinates (57) satisfy equations analogous to (37) and (38).

In like manner the four surfaces whose coördinates are of the form

$$x/\theta_1, \qquad -\lambda_1 x_1, \qquad -x_2/\theta_2', \qquad \lambda_2' x'$$

form a quatern in radial relation to the corresponding surfaces of the other two quaterns.

Accordingly we have

THEOREM 9. *When a quatern of surfaces is known, two other quaterns can be found without quadratures, each of which consists of two pairs of associate surfaces.*

6. W-CONGRUENCES ASSOCIATED WITH A TRANSFORMATION K

We consider a surface S and the associate surface S_0 whose coördinates x_0, y_0, z_0 are given by quadratures of the form

$$\frac{\partial x_0}{\partial u} = -\rho\frac{\partial x}{\partial u}, \qquad \frac{\partial x_0}{\partial v} = \rho\frac{\partial x}{\partial v}, \tag{59}$$

as follows from (11). These coördinates are solutions of

$$\frac{\partial^2\theta_0}{\partial u\,\partial v} = \frac{\partial\log\sqrt{\rho}}{\partial v}\frac{\partial\theta_0}{\partial u} + \frac{\partial\log\sqrt{\rho}}{\partial u}\frac{\partial\theta_0}{\partial v}, \tag{60}$$

as is readily shown, if we note that x is a solution of equation (1). If we put

$$\sigma = \theta_0/\sqrt{\rho}, \tag{61}$$

the function σ is a solution of the equation

$$\frac{\partial^2\sigma}{\partial u\,\partial v} = \sqrt{\rho}\,\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sqrt{\rho}}\right)\sigma. \tag{62}$$

In accordance with the theory of infinitesimal deformation of surfaces the surface Σ, whose coördinates ξ, η, ζ are given by quadratures of the form

$$\frac{\partial \xi}{\partial u} = z_0 \frac{\partial y}{\partial u} - y_0 \frac{\partial z}{\partial u}, \qquad \frac{\partial \xi}{\partial v} = z_0 \frac{\partial y}{\partial v} - y_0 \frac{\partial z}{\partial v}, \tag{63}$$

corresponds to S with orthogonality of linear elements.* For the sake of brevity we say that Σ is the *ortho-surface* of S determined by S_0.

By means of (59), equations (63) are reducible to

$$\frac{\partial \xi}{\partial u} = -\frac{1}{\rho}\left(z_0 \frac{\partial y_0}{\partial u} - y_0 \frac{\partial z_0}{\partial u} \right), \qquad \frac{\partial \xi}{\partial v} = \frac{1}{\rho}\left(z_0 \frac{\partial y_0}{\partial v} - y_0 \frac{\partial z_0}{\partial v} \right). \tag{64}$$

If we put

$$x_0 = \sqrt{\rho}\,\alpha, \qquad y_0 = \sqrt{\rho}\,\beta, \qquad z_0 = \sqrt{\rho}\,\gamma, \tag{65}$$

the functions α, β, γ are solutions of equation (62) and equations (64) assume the Lelieuvre form

$$\frac{\partial \xi}{\partial u} = \left(\beta \frac{\partial \gamma}{\partial u} - \gamma \frac{\partial \beta}{\partial u} \right), \qquad \frac{\partial \xi}{\partial v} = -\left(\beta \frac{\partial \gamma}{\partial v} - \gamma \frac{\partial \beta}{\partial v} \right). \tag{66}$$

In like manner there is a surface S_{10} associate to a surface S_1 resulting from S by a transformation K and its coördinates, x_{10}, y_{10}, z_{10}, are given by

$$\frac{\partial x_{10}}{\partial u} = -\rho_1 \frac{\partial x_1}{\partial u}, \qquad \frac{\partial x_{10}}{\partial v} = \rho_1 \frac{\partial x_1}{\partial v}, \tag{67}$$

where, as follows from (29),

$$\rho_1 = \lambda^2/\rho\theta^2. \tag{68}$$

It can readily be shown that the four surfaces S, S_0, S_1, S_{10} satisfy the requirements of Theorem 5, so that S_0 and S_{10} are in the relation of a transformation K. In fact, in this case equation (40) becomes

$$\theta(x_{10} - x_0) = \lambda(x_1 - x). \tag{69}$$

If the expressions (10) for $\partial x_1/\partial u$ and $\partial x_1/\partial v$ be substituted in (67) and $(x_1 - x)$ be replaced by its value from (69), we have

$$\frac{\partial}{\partial u}(\theta x_{10}) = x_0 \frac{\partial \theta}{\partial u} + \frac{\lambda}{\rho}\frac{\partial x_0}{\partial u}, \qquad \frac{\partial}{\partial v}(\theta x_{10}) = x_0 \frac{\partial \theta}{\partial v} - \frac{\lambda}{\rho}\frac{\partial x_0}{\partial v}. \tag{70}$$

In a manner analogous to (65) we put

$$x_{10} = \sqrt{\rho_1}\,\alpha_1 = -\frac{\lambda}{\sqrt{\rho}\,\theta}\alpha_1. \tag{71}$$

As defined by equations (8), the function λ is a solution of equation (60).

* E., p. 382.

Consequently the function σ_1, defined by

$$\sigma_1 = \lambda / \sqrt{\rho}, \tag{72}$$

is a solution of (62). In consequence of equations (71), (72), and (8), equations (70) can be written in the form

$$\frac{\partial}{\partial u}(\sigma_1 \alpha_1) = -\left(\sigma_1 \frac{\partial \alpha}{\partial u} - \alpha \frac{\partial \sigma_1}{\partial u}\right), \qquad \frac{\partial}{\partial v}(\sigma_1 \alpha_1) = \left(\sigma_1 \frac{\partial \alpha}{\partial v} - \alpha \frac{\partial \sigma_1}{\partial v}\right). \tag{73}$$

Moreover, the coördinates ξ_1, η_1, ζ, of Σ_1, the ortho-surface to S_1 determined by S_{10}, are given by the quadratures

$$\frac{\partial \xi_1}{\partial u} = \left(\beta_1 \frac{\partial \gamma_1}{\partial u} - \gamma_1 \frac{\partial \beta_1}{\partial u}\right), \qquad \frac{\partial \xi_1}{\partial v} = -\left(\beta_1 \frac{\partial \gamma_1}{\partial v} - \gamma_1 \frac{\partial \beta_1}{\partial v}\right). \tag{74}$$

From these results we know that Σ and Σ_1 can be so placed in space that they are the focal surfaces of a W-congruence, and that the following relations hold:

$$\begin{gathered} \xi_1 - \xi = \beta\gamma_1 - \beta_1 \gamma, \qquad \eta_1 - \eta = \gamma\alpha_1 - \gamma_1 \alpha, \\ \zeta_1 - \zeta = \alpha\beta_1 - \alpha_1 \beta. \end{gathered} \tag{75}$$

Each solution of equation (62) determines a surface associate to Σ. We denote by Σ_{01} the associate surface corresponding to the solution σ_1 given by (72), and by Σ_{02} the associate surface determined by the solution

$$\sigma_2 = 1 / \sqrt{\rho}. \tag{76}$$

The first of these gives rise to the W-congruence whose focal surfaces are Σ and Σ_1. The surface Σ_{01} determines an ortho-surface to Σ, namely $\bar{S}$, whose coördinates $\bar{x}, \bar{y}, \bar{z}$ are

$$\bar{x} = \sigma_1 \alpha_1, \qquad \bar{y} = \sigma_1 \beta_1, \qquad \bar{z} = \sigma_1 \gamma_1, \tag{77}$$

where $\alpha_1, \beta_1, \gamma_1$ are given by (71). The surface $\bar{S}_0$ whose coördinates are

$$\bar{x}_0 = \alpha / \sigma_1, \qquad \bar{y}_0 = \beta / \sigma_1, \qquad \bar{z}_0 = \gamma / \sigma_1, \tag{78}$$

is associate to $\bar{S}$ and is an ortho-surface of Σ_1.†

The equation for Σ_1 analogous to (62), namely

$$\frac{\partial^2 \bar{\sigma}}{\partial u \, \partial v} = \sqrt{\rho_1} \frac{\partial^2}{\partial u \, \partial v}\left(\frac{1}{\sqrt{\rho_1}}\right)\bar{\sigma}, \tag{79}$$

admits the solution $1/\sigma_1$. This solution determines an associate surface of Σ_1 which leads to the ortho-surface $\bar{S}_0$ of Σ_1. Moreover, the joins of corresponding points on this associate surface and Σ_{01} are concurrent.‡

* Cf. E., pp. 417–420.

† Cf. E., p. 420.

‡ Cf. Darboux, *Leçons*, vol. 4, p. 69.

We observe that $1/\sqrt{\rho_1}$ also is a solution of equation (79). We denote by Σ_{10} the surface associate to Σ_1 determined by this solution. If we apply the Moutard transformation of the form (73) to $1/\sqrt{\rho}$, we obtain $1/\sqrt{\rho_1}$. In fact, it is readily shown that the equations

$$\begin{aligned}\frac{\partial}{\partial u}\left(\sigma_1\frac{1}{\sqrt{\rho_1}}\right) &= -\left(\sigma_1\frac{\partial}{\partial u}\frac{1}{\sqrt{\rho}} - \frac{1}{\sqrt{\rho}}\frac{\partial\sigma_1}{\partial u}\right),\\ \frac{\partial}{\partial v}\left(\sigma_1\frac{1}{\sqrt{\rho_1}}\right) &= \left(\sigma_1\frac{\partial}{\partial v}\frac{1}{\sqrt{\rho}} - \frac{1}{\sqrt{\rho}}\frac{\partial\sigma_1}{\partial v}\right),\end{aligned} \tag{80}$$

reduce to equations (8). Making use of results which we have established elsewhere,* we have

THEOREM 10. *The surfaces Σ_{02} and Σ_{10}, associate to Σ and Σ_1, and determined by $1/\sqrt{\rho}$ and $1/\sqrt{\rho_1}$ respectively, are such that the lines joining corresponding points on these surfaces form a congruence whose developables meet them in the conjugate parametric system, which has equal tangential invariants; and the lines of intersection of the tangent planes to Σ_{02} and Σ_{10} form a congruence whose developables are parametric and whose focal planes are harmonic to these tangent planes.*

We say that two surfaces, related as Σ_{02} and Σ_{10} are in this theorem, are in the relation of a *transformation* Ω.

We turn to the consideration of the W-congruence, determined by the solution $\sigma_2 = 1/\sqrt{\rho}$ of equation (62), of which Σ is one of the focal surfaces. Since S is the corresponding ortho-surface of Σ, the functions α_2, β_2, γ_2 for the second sheet Σ_2 are given by the equations

$$\alpha_2 = \sqrt{\rho}\,x, \qquad \beta_2 = \sqrt{\rho}\,y, \qquad \gamma_2 = \sqrt{\rho}\,z, \tag{81}$$

which are analogous to equations (77). These functions are solutions of the equation

$$\frac{\partial^2\sigma'}{\partial u\,\partial v} = \frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial u\,\partial v}\cdot\sigma', \tag{82}$$

as is readily shown. Moreover, from (59), (65), and (81) it follows that

$$\begin{aligned}\frac{\partial}{\partial u}\left(\frac{\alpha_2}{\sqrt{\rho}}\right) &= -\left(\frac{1}{\sqrt{\rho}}\frac{\partial\alpha}{\partial u} - \alpha\frac{\partial}{\partial u}\frac{1}{\sqrt{\rho}}\right),\\ \frac{\partial}{\partial v}\left(\frac{\alpha_2}{\sqrt{\rho}}\right) &= \left(\frac{1}{\sqrt{\rho}}\frac{\partial\alpha}{\partial v} - \alpha\frac{\partial}{\partial v}\frac{1}{\sqrt{\rho}}\right),\end{aligned} \tag{83}$$

which are of the same form as (73).

In like manner the function $\bar{\sigma}_1 = 1/\sqrt{\rho_1}$ determines a W-congruence for

*Rendiconti, loc. cit.

which Σ_1 is one of the focal surfaces. Since S_1 is the corresponding orthosurface to Σ_1, the functions α', β', γ' for the second focal sheet Σ' are given by

$$\alpha' = \sqrt{\rho_1}\, x_1 = -\frac{\lambda}{\sqrt{\rho}\,\theta} x_1, \qquad \beta' = -\frac{\lambda}{\sqrt{\rho}\,\theta} y_1, \qquad \gamma' = -\frac{\lambda}{\sqrt{\rho}\,\theta} z_1. \tag{84}$$

It is easily found that the function

$$\sigma_2' = \sqrt{\rho}\,\theta \tag{85}$$

is a solution of equation (82). If we substitute in equations (10) the expressions for x_1 and x which follow from equations (81) and (84), the resulting equations may be given the form

$$\begin{aligned} \frac{\partial}{\partial u}(\sigma_2'\,\alpha') &= -\left(\sigma_2' \frac{\partial \alpha_2}{\partial u} - \alpha_2 \frac{\partial \sigma_2'}{\partial u}\right), \\ \frac{\partial}{\partial v}(\sigma_2'\,\alpha') &= \left(\sigma_2' \frac{\partial \alpha_2}{\partial v} - \alpha_2 \frac{\partial \sigma_2'}{\partial v}\right). \end{aligned} \tag{86}$$

Comparing these equations with (73), we observe that the lines joining corresponding points on the surfaces Σ_2 and Σ' form a W-congruence, for which these are the focal surfaces. It remains to find the relations between the coördinates of these surfaces.

From equations (65), (69), (71), (81), and (84), we obtain the relation

$$\alpha' = \alpha + \frac{\lambda}{\rho\theta}(\alpha_1 - \alpha_2). \tag{87}$$

Analogous to equations (75) are the following:

$$\begin{aligned} \xi_2 - \xi = \beta\gamma_2 - \gamma\beta_2, \qquad \xi' - \xi_1 &= \beta_1\gamma' - \gamma_1\beta', \\ \xi' - \xi_2 &= \beta_2\gamma' - \gamma_2\beta', \end{aligned} \tag{88}$$

which are consistent in view of (87). From these follows also

$$\xi' = \xi + \frac{\lambda}{\rho\theta}(\beta_2\gamma_1 - \beta_1\gamma_2). \tag{89}$$

From the foregoing results we have

THEOREM 11. *When a pair of surfaces S and S_1 in the relation of a transformation K are known, it is possible to find by quadratures four surfaces Σ, Σ_1, Σ_2, Σ' such that the pairs $\Sigma\Sigma_1$, $\Sigma\Sigma_2$, $\Sigma_1\Sigma'$, $\Sigma_2\Sigma'$ are the focal surfaces of four W-congruences and the asymptotic lines on these surfaces correspond.*

We say that four such W-congruences form a *quatern*, which we indicate by $W(\Sigma, \Sigma_1, \Sigma_2, \Sigma')$.

From this theorem we derive

THEOREM 12. *Four surfaces S, S_1, S_2, S' forming a quatern under transformations K determine twelve W-congruences forming six quaterns of W-congruences.*

Let $W(\Sigma, \Sigma_1, \Sigma_3, \Sigma')$ and $W(\Sigma, \Sigma_2, \Sigma_3, \Sigma'')$ be the quaterns determined by S and S_1 and by S and S_2 respectively in accordance with Theorem 11. The pair of surfaces S_1 and S' determine a quatern $W(\Sigma_1, \Sigma_4, \Sigma', \Sigma''')$. But from the nature of the theorem of permutability it follows that there is a quatern $W(\Sigma_2, \Sigma_4, \Sigma'', \Sigma''')$ determined by S_2 and S'. Furthermore, it is readily seen that these focal surfaces are so placed that there exist also the quaterns $W(\Sigma, \Sigma_1, \Sigma_2, \Sigma_4)$ and $W(\Sigma_3, \Sigma', \Sigma'', \Sigma''')$.

7. TRANSFORMATIONS K DETERMINED BY A QUATERN OF W-CONGRUENCES

In this section we shall show that every quatern of W-congruences gives rise to four pairs of surfaces in the relation of transformations K. In doing so we make use of the results of Bianchi* concerning the determination of such quaterns.

Let Σ be a surface referred to its asymptotic lines and defined by equations (66), where α, β, γ are linearly independent solutions of an equation of the form

$$\frac{\partial^2 \sigma}{\partial u\, \partial v} = M\sigma, \tag{90}$$

M being a function of u and v in general. If σ_1 and σ_2 are two independent solutions of this equation, the functions α_1, β_1, γ_1 and α_2, β_2, γ_2, given by quadratures of the form (73), determine two surfaces Σ_1 and Σ_2 whose cartesian coördinates ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 are expressible in the form (75). Moreover, the functions α_i, β_i, γ_i $(i = 1, 2)$ satisfy the equations

$$\frac{\partial^2 \bar{\sigma}_i}{\partial u\, \partial v} = \sigma_i \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sigma_i}\right)\bar{\sigma}_i \qquad (i = 1, 2). \tag{91}$$

In accordance with the theorem of Moutard these equations are satisfied also by the functions σ_1' and σ_2', defined by

$$\frac{\partial}{\partial u}(\sigma_1 \sigma_1') = -\sigma_1^2 \frac{\partial}{\partial u}\left(\frac{\sigma_2}{\sigma_1}\right), \qquad \frac{\partial}{\partial v}(\sigma_1 \sigma_1') = \sigma_1^2 \frac{\partial}{\partial v}\left(\frac{\sigma_2}{\sigma_1}\right), \tag{92}$$

$$\frac{\partial}{\partial u}(\sigma_2 \sigma_2') = -\sigma_2^2 \frac{\partial}{\partial u}\left(\frac{\sigma_1}{\sigma_2}\right), \qquad \frac{\partial}{\partial v}(\sigma_2 \sigma_2') = \sigma_2^2 \frac{\partial}{\partial v}\left(\frac{\sigma_1}{\sigma_2}\right). \tag{93}$$

In the subsequent discussion we assume that the constants of integration are so chosen that

$$\sigma_2 \sigma_2' = -\sigma_1 \sigma_1', \tag{94}$$

* *Lezioni di Geometria Differenziale*, Pisa (1903), vol. 2, pp. 71–74.

in which case we have

$$\sigma_1' \frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sigma_1'}\right) = \sigma_2' \frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sigma_2'}\right). \tag{95}$$

Bianchi has shown that the functions α', β', γ', defined by

$$\begin{gathered}\alpha' = \alpha + t(\alpha_1 - \alpha_2), \qquad \beta' = \beta + t(\beta_1 - \beta_2), \\ \gamma' = \gamma + t(\gamma_1 - \gamma_2),\end{gathered} \tag{96}$$

where

$$t = -\frac{\sigma_2}{\sigma_1'} = \frac{\sigma_1}{\sigma_2'}, \tag{97}$$

satisfy equations (86), also the equations

$$\begin{aligned}\frac{\partial}{\partial u}(\sigma_1'\,\alpha') &= -\left(\sigma_1' \frac{\partial \alpha_1}{\partial u} - \alpha_1 \frac{\partial \sigma_1'}{\partial u}\right), \\ \frac{\partial}{\partial v}(\sigma_1'\,\alpha') &= \left(\sigma_1' \frac{\partial \alpha_1}{\partial v} - \alpha_1 \frac{\partial \sigma_1'}{\partial v}\right),\end{aligned} \tag{98}$$

and analogous ones for β' and γ', and finally also

$$\frac{\partial^2 \theta'}{\partial u\,\partial v} = M'\,\theta', \tag{99}$$

where M' is the function equal to either member of (95).

These functions α', β', γ' determine a surface Σ', whose coördinates are given by

$$\begin{gathered}\xi' = \xi + t(\beta_2\,\gamma_1 - \beta_1\,\gamma_2), \qquad \eta' = \eta + t(\gamma_2\,\alpha_1 - \gamma_1\,\alpha_2), \\ \zeta' = \zeta + t(\alpha_2\,\beta_1 - \alpha_1\,\beta_2),\end{gathered} \tag{100}$$

such that Σ_1 and Σ' are the focal surfaces of a W-congruence and likewise Σ_2 and Σ' of another W-congruence. Hence we have a quatern of W-congruences $W(\Sigma, \Sigma_1, \Sigma_2, \Sigma')$.

We suppose that we have a quatern $W(\Sigma, \Sigma_1, \Sigma_2, \Sigma')$ defined in this general way and we introduce three functions ρ, λ, θ, defined by the equations

$$\sigma_1 = \lambda/\sqrt{\rho}, \qquad \sigma_2 = 1/\sqrt{\rho}, \qquad \sigma_2' = \sqrt{\rho}\,\theta. \tag{101}$$

In terms of these functions equation (90) can be given the form (62) and equations (93) reduce to (8). In accordance with (94) we have

$$\sigma_1' = -\theta\sqrt{\rho}/\lambda = 1/\sqrt{\rho_1}, \tag{102}$$

the function $\sqrt{\rho_1}$ being defined by this equation.

In accordance with the general theory of W-congruences and infinitesimal

deformation of surfaces, the surfaces S and S_0 whose coördinates x, y, z and x_0, y_0, z_0 are given by (81) and (65) are ortho-surfaces of Σ and Σ_2 respectively. In like manner the surfaces $\bar{S}$ and $\bar{S}_0$ whose coördinates $\bar{x}, \bar{y}, \bar{z}$ and $\bar{x}_0, \bar{y}_0, \bar{z}_0$ are of the form (77) and (78) are ortho-surfaces of Σ and Σ_1; and S_1 and S_{10}, whose coördinates x_1, y_1, z_1 and x_{10}, y_{10}, z_{10} are given by (84) and (71), are ortho-surfaces of Σ_1 and Σ'. Also the quantities $\bar{x}_1, \bar{y}_1, \bar{z}_1$ and $\bar{x}_{10}, \bar{y}_{10}, \bar{z}_{10}$ which are of the form

$$\bar{x}_1 = \sqrt{\rho}\,\theta\alpha', \qquad \bar{x}_{10} = \alpha_2/\sqrt{\rho}\,\theta \tag{103}$$

are the coördinates of ortho-surfaces $\bar{S}_1$ and $\bar{S}_{10}$ of Σ_2 and Σ' respectively.

When the values of α_2 and α' given by (81) and (84) are substituted in (86), we obtain equations (4). Hence the surfaces S and S_1 are in the relation of a transformation K. In like manner it can be shown that each pair of surfaces $S\bar{S}_1$, $\bar{S}_0\bar{S}_{10}$, S_0S_{10} are in the relation of a transformation K. Moreover, it follows from Theorem 10 that each of the above four pairs determines a pair of surfaces referred to a conjugate system with equal tangential invariants which are in the relation of a transformation Ω, and the spherical representation of this conjugate system on any one of these surfaces is the same as of the asymptotic lines on one of the surfaces $\Sigma, \Sigma_1, \Sigma_2, \Sigma'$. Hence we have

THEOREM 13. *If $\Sigma, \Sigma_1, \Sigma_2, \Sigma'$ are the focal surfaces of a quatern of W-congruences, the eight associated ortho-surfaces of the former can be arranged into four pairs such that the surfaces of each pair are in the relation of a transformation K; moreover, each pair determines the quatern of W-congruences; also each pair determines two other surfaces which are in the relation of a transformation Ω.*

8. EQUATIONS OF A TRANSFORMATION K IN ANOTHER FORM

In applying the transformations K to particular types of surfaces whose transforms are to be of the same type, it is frequently advisable to have the equations of the transformation in another form, which we will now establish.

Let S be a surface referred to a conjugate system with equal point invariants, and let X, Y, Z; X_1, Y_1, Z_1 and X_2, Y_2, Z_2, denote the direction-cosines of the normal to S and of the bisectors of the angles between the coördinate lines. If one of these angles be denoted by 2ω, we have

$$\frac{\partial x}{\partial u} = \sqrt{E}\,(\cos\omega X_1 - \sin\omega X_2), \qquad \frac{\partial x}{\partial v} = \sqrt{G}\,(\cos\omega X_1 + \sin\omega X_2), \tag{104}$$

where E, F, G denote the coefficients of the linear element of S. Since

$F = \sqrt{EG}\cos 2\omega$, we have

$$\begin{aligned}\frac{\partial X}{\partial u} &= -\frac{D}{\sqrt{E}}\frac{1}{\sin 2\omega}(\sin\omega X_1 - \cos\omega X_2),\\ \frac{\partial X}{\partial v} &= \frac{D''}{\sqrt{G}}\frac{1}{\sin 2\omega}(\sin\omega X_1 + \cos\omega X_2),\end{aligned} \tag{105}$$

where D and D'' denote the fundamental coefficients of the second order for S.*

If we compare equation (1) with the Gauss equations for S†, we observe that

$$\left\{\begin{matrix}12\\1\end{matrix}\right\} = -\frac{\partial \log\sqrt{\rho}}{\partial v}, \qquad \left\{\begin{matrix}12\\2\end{matrix}\right\} = -\frac{\partial \log\sqrt{\rho}}{\partial u}, \tag{106}$$

the Christoffel symbols $\left\{\begin{matrix}rs\\t\end{matrix}\right\}$ being formed with respect to the linear element of S. From the definition of these symbols‡ it follows that equations (106) lead to the identities

$$\begin{aligned}\frac{\partial\sqrt{E}}{\partial v} &= -\sqrt{E}\frac{\partial\log\sqrt{\rho}}{\partial v} - \sqrt{G}\cos 2\omega\frac{\partial\log\sqrt{\rho}}{\partial u},\\ \frac{\partial\sqrt{G}}{\partial u} &= -\sqrt{G}\frac{\partial\log\sqrt{\rho}}{\partial u} - \sqrt{E}\cos 2\omega\frac{\partial\log\sqrt{\rho}}{\partial v}.\end{aligned} \tag{107}$$

The expressions for the other Christoffel symbols are as follows:

$$\begin{aligned}\left\{\begin{matrix}11\\1\end{matrix}\right\} &= \frac{\partial\log\sqrt{E}}{\partial u} + 2\cot 2\omega\frac{\partial\omega}{\partial u} - \sqrt{\frac{E}{G}}\cos 2\omega\frac{\partial\log\sqrt{\rho}}{\partial v},\\ \left\{\begin{matrix}11\\2\end{matrix}\right\} &= \frac{E}{G}\frac{\partial\log\sqrt{\rho}}{\partial v} - 2\sqrt{\frac{E}{G}}\frac{1}{\sin 2\omega}\frac{\partial\omega}{\partial u},\\ \left\{\begin{matrix}22\\1\end{matrix}\right\} &= \frac{G}{E}\frac{\partial\log\sqrt{\rho}}{\partial u} - 2\sqrt{\frac{G}{E}}\frac{1}{\sin 2\omega}\frac{\partial\omega}{\partial v},\\ \left\{\begin{matrix}22\\2\end{matrix}\right\} &= \frac{\partial\log\sqrt{G}}{\partial v} + 2\cot 2\omega\frac{\partial\omega}{\partial v} - \sqrt{\frac{G}{E}}\cos 2\omega\frac{\partial\log\sqrt{\rho}}{\partial u}.\end{aligned} \tag{108}$$

If equations (104) be differentiated and use be made of equation (1) and

* E., p. 116.
† E., p. 154.
‡ E., p. 153.

of the Gauss equations*

$$\begin{aligned}\frac{\partial^2 x}{\partial u^2} &= \left\{\begin{matrix}11\\1\end{matrix}\right\}\frac{\partial x}{\partial u} + \left\{\begin{matrix}11\\2\end{matrix}\right\}\frac{\partial x}{\partial v} + DX,\\ \frac{\partial^2 x}{\partial v^2} &= \left\{\begin{matrix}22\\1\end{matrix}\right\}\frac{\partial x}{\partial u} + \left\{\begin{matrix}22\\2\end{matrix}\right\}\frac{\partial x}{\partial v} + D''X,\end{aligned} \tag{109}$$

we obtain

$$\begin{aligned}\frac{\partial X_1}{\partial u} &= \frac{D}{2\sqrt{E}\cos\omega}X + \left(\sqrt{\frac{E}{G}}\frac{\partial \log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}\right)X_2,\\ \frac{\partial X_1}{\partial v} &= \frac{D''}{2\sqrt{G}\cos\omega}X - \left(\sqrt{\frac{G}{E}}\frac{\partial \log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}\right)X_2,\\ \frac{\partial X_2}{\partial u} &= \frac{-D}{2\sqrt{E}\sin\omega}X - \left(\sqrt{\frac{E}{G}}\frac{\partial \log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}\right)X_1,\\ \frac{\partial X_2}{\partial v} &= \frac{D''}{2\sqrt{G}\sin\omega}X + \left(\sqrt{\frac{G}{E}}\frac{\partial \log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}\right)X_1.\end{aligned} \tag{110}$$

The conditions of integrability of these equations are satisfied in consequence of the Gauss and Codazzi equations† which in the present case can be given the form

$$\begin{aligned}\frac{DD''}{\sqrt{EG}}\frac{1}{\sin 2\omega} &= \frac{\partial}{\partial v}\left(\sqrt{\frac{E}{G}}\sin 2\omega\,\frac{\partial \log\sqrt{\rho}}{\partial v}\right)\\ &\quad + \frac{\partial}{\partial u}\left(\sqrt{\frac{G}{E}}\sin 2\omega\,\frac{\partial \log\sqrt{\rho}}{\partial u}\right),\end{aligned} \tag{111}$$

and

$$\begin{aligned}\frac{\partial D}{\partial v} + \frac{\partial \log\sqrt{\rho}}{\partial v}D + \left(\frac{E}{G}\frac{\partial \log\sqrt{\rho}}{\partial v} - 2\sqrt{\frac{E}{G}}\frac{1}{\sin 2\omega}\frac{\partial\omega}{\partial u}\right)D'' &= 0,\\ \frac{\partial D''}{\partial u} + \frac{\partial \log\sqrt{\rho}}{\partial u}D'' + \left(\frac{G}{E}\frac{\partial \log\sqrt{\rho}}{\partial u} - 2\sqrt{\frac{G}{E}}\frac{1}{\sin 2\omega}\frac{\partial\omega}{\partial v}\right)D &= 0.\end{aligned} \tag{112}$$

We introduce three functions a, b, w by means of equations of the form

$$x_1 - x = \frac{1}{\lambda}(aX_1 + bX_2 + wX), \tag{113}$$

where x_1, y_1, z_1 are the cartesian coördinates of a surface S_1 in the relation of a transformation K with S and λ is the corresponding function defined by

* E., p. 154.
† E., p. 155.

equations (8). In consequence of (113), the fundamental equations (10) can be written

$$\frac{\partial x_1}{\partial u} = \frac{\rho}{\lambda^2}\Big[\Big(a\frac{\partial\theta}{\partial u} + \theta\lambda\cos\omega\sqrt{E}\Big)X_1 + \Big(b\frac{\partial\theta}{\partial u} - \theta\lambda\sin\omega\sqrt{E}\Big)X_2 + w\frac{\partial\theta}{\partial u}X\Big],$$

(114)

$$\frac{\partial x_1}{\partial v} = -\frac{\rho}{\lambda^2}\Big[\Big(a\frac{\partial\theta}{\partial v} + \theta\lambda\cos\omega\sqrt{G}\Big)X_1 + \Big(b\frac{\partial\theta}{\partial v} + \theta\lambda\sin\omega\sqrt{G}\Big)X_2 + w\frac{\partial\theta}{\partial v}X\Big].$$

When the expression (113) for x_1 is substituted in these equations, we obtain two equations of the form

$$A_1X_1 + B_1X_2 + C_1X = 0, \qquad A_2X_1 + B_2X_2 + C_2X = 0,$$

where $A_1, B_1, \cdots, C_2$ are determinate functions which must be zero, since these equations are satisfied also by the Y's and Z's. This gives the following six equations of condition:

$$\frac{\partial a}{\partial u} = (\rho\theta - \lambda)\sqrt{E}\cos\omega + b\Big(\sqrt{\frac{E}{G}}\frac{\partial\log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}\Big) + \frac{wD}{2\sqrt{E}\cos\omega},$$

$$\frac{\partial a}{\partial v} = -(\rho\theta + \lambda)\sqrt{G}\cos\omega - b\Big(\sqrt{\frac{G}{E}}\frac{\partial\log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}\Big) + \frac{wD''}{2\sqrt{G}\cos\omega},$$

(115)

$$\frac{\partial b}{\partial u} = -(\rho\theta - \lambda)\sqrt{E}\sin\omega - a\Big(\sqrt{\frac{E}{G}}\frac{\partial\log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}\Big) - \frac{wD}{2\sqrt{E}\sin\omega},$$

$$\frac{\partial b}{\partial v} = -(\rho\theta + \lambda)\sqrt{G}\sin\omega + a\Big(\sqrt{\frac{G}{E}}\frac{\partial\log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}\Big) + \frac{wD''}{2\sqrt{G}\sin\omega},$$

$$\frac{\partial w}{\partial u} = -\frac{D}{2\sqrt{E}}\Big(\frac{a}{\cos\omega} - \frac{b}{\sin\omega}\Big), \qquad \frac{\partial w}{\partial v} = -\frac{D''}{2\sqrt{G}}\Big(\frac{a}{\cos\omega} + \frac{b}{\sin\omega}\Big).$$

With the aid of equations (8), (107), (111), and (112) it can be shown that the conditions of integrability of these equations are satisfied.

If we put for the sake of brevity

$$T^2 = a^2 + b^2 + w^2, \tag{116}$$

it follows from (115) that

$$\begin{aligned} T\frac{\partial T}{\partial u} &= (\rho\theta - \lambda)\sqrt{E}\,(a\cos\omega - b\sin\omega), \\ T\frac{\partial T}{\partial v} &= -(\rho\theta + \lambda)\sqrt{G}\,(a\cos\omega + b\sin\omega). \end{aligned} \tag{117}$$

If X', Y', Z' denote the direction-cosines of the normal to S_1, we have from (114),

$$\begin{aligned} \Delta X' = &- w\sin\omega\left(\sqrt{E}\frac{\partial\theta}{\partial v} + \sqrt{G}\frac{\partial\theta}{\partial u}\right)X_1 \\ &+ w\cos\omega\left(-\sqrt{E}\frac{\partial\theta}{\partial v} + \sqrt{G}\frac{\partial\theta}{\partial u}\right)X_2 \\ &+ X\left[a\sin\omega\left(\sqrt{E}\frac{\partial\theta}{\partial v} + \sqrt{G}\frac{\partial\theta}{\partial u}\right)\right. \\ &\left. + b\cos\omega\left(\sqrt{E}\frac{\partial\theta}{\partial v} - \sqrt{G}\frac{\partial\theta}{\partial u}\right) + \theta\lambda\sqrt{EG}\sin 2\omega\right], \end{aligned} \tag{118}$$

where Δ^2 is equal to the sum of the squares of the coefficients of X_1, X_2, and X.

9. When S and S_1 envelop a two-parameter family of spheres. Transformations D_m of an isothermic surface

As an example of the foregoing method we consider the case where S and S_1 constitute the envelope of a two parameter family of spheres. In this case we must have

$$\begin{aligned} x_1 - x = \frac{R\Delta}{\lambda}(X' - X), \qquad y_1 - y = \frac{R\Delta}{\lambda}(Y' - Y), \\ z_1 - z = \frac{R\Delta}{\lambda}(Z' - Z), \end{aligned} \tag{119}$$

where $R\Delta/\lambda$ is the radius of the spheres.

When the expressions for x_1 and X' given by (113) and (118) are substituted in the first of equations (119), and the coefficients of X_1, X_2, and X on

the two sides of the resulting identity are equated, we get

$$
\begin{aligned}
a &= -w \sin\omega R\left(\sqrt{E}\frac{\partial\theta}{\partial v}+\sqrt{G}\frac{\partial\theta}{\partial u}\right),\\
b &= -w\cos\omega R\left(\sqrt{E}\frac{\partial\theta}{\partial v}-\sqrt{G}\frac{\partial\theta}{\partial u}\right),\\
w &= \left[a\sin\omega\left(\sqrt{E}\frac{\partial\theta}{\partial v}+\sqrt{G}\frac{\partial\theta}{\partial u}\right)+b\cos\omega\left(\sqrt{E}\frac{\partial\theta}{\partial v}-\sqrt{G}\frac{\partial\theta}{\partial u}\right)\right.\\
&\qquad\left.+\theta\lambda\sqrt{EG}\sin 2\omega-\Delta\right]R.
\end{aligned}
\tag{120}
$$

By means of the first two of these equations the last is equivalent to

$$a^2+b^2+w^2=Rw(\theta\lambda\sqrt{EG}\sin 2\omega-\Delta). \tag{121}$$

In consequence of the first two of equations (120) equation (118) can be written

$$\Delta X'=\frac{a}{R}X_1+\frac{b}{R}X_2-X\left[\frac{a^2+b^2}{Rw}-\theta\lambda\sqrt{EG}\sin 2\omega\right], \tag{122}$$

whence it follows that

$$\Delta^2=\frac{a^2}{R^2}+\frac{b^2}{R^2}+\left(\frac{a^2+b^2}{Rw}-\theta\lambda\sqrt{EG}\sin 2\omega\right)^2. \tag{123}$$

Eliminating Δ from (121) and (123), we obtain

$$a^2+b^2+w^2=2w\theta\lambda\sqrt{EG}\sin 2w\,R, \tag{124}$$

and consequently from (121) and (124) we have

$$\Delta=-\theta\lambda\sqrt{EG}\sin 2\omega. \tag{125}$$

By means of (120), (121), (125) we obtain the following equations:

$$
\begin{aligned}
\sqrt{E}(a\cos\omega-b\sin\omega)&=-w\sin 2\omega\sqrt{EG}\frac{\partial\theta}{\partial u}R=-\frac{T^2}{2\theta\lambda}\frac{\partial\theta}{\partial u},\\
\sqrt{G}(a\cos\omega+b\sin\omega)&=-w\sin 2\omega\sqrt{EG}\frac{\partial\theta}{\partial v}R=-\frac{T^2}{2\theta\lambda}\frac{\partial\theta}{\partial v}.
\end{aligned}
\tag{126}
$$

With the aid of these expressions and (8) equations (117) are reducible to

$$
\begin{aligned}
\frac{\partial\log T^2}{\partial u}&=\left(\frac{1}{\theta}-\frac{\rho}{\lambda}\right)\frac{\partial\theta}{\partial u}=\frac{\partial}{\partial u}\log(\lambda\theta),\\
\frac{\partial\log T^2}{\partial v}&=\left(\frac{1}{\theta}+\frac{\rho}{\lambda}\right)\frac{\partial\theta}{\partial v}=\frac{\partial}{\partial v}\log(\lambda\theta).
\end{aligned}
$$

Hence on integration we have

$$T^2 = 2c\lambda\theta, \tag{127}$$

where c denotes an arbitrary constant.

Substituting in (126) the value (127) of T^2, we get

$$\begin{aligned} \sqrt{E}\,(a\cos\omega - b\sin\omega) &= -c\,\frac{\partial\theta}{\partial u}, \\ \sqrt{G}\,(a\cos\omega + b\sin\omega) &= -c\,\frac{\partial\theta}{\partial v}. \end{aligned} \tag{128}$$

If these equations be differentiated with respect to v and u respectively, and in the reduction use be made of the preceding equations, we find

$$(\lambda + \rho\theta)\sqrt{G}\cos 2\omega = 0, \qquad (\lambda - \rho\theta)\sqrt{E}\cos 2\omega = 0.$$

Hence we must have $2\omega = \pi/2$, that is, the parametric curves are the lines of curvature. From (107) it follows that

$$\sqrt{E} = V/\sqrt{\rho}, \qquad \sqrt{G} = U/\sqrt{\rho}, \tag{129}$$

where U and V are functions of u and v alone respectively, and consequently S is an isothermic surface. As the relation between S and S_1 is reciprocal, it follows that S_1 also is isothermic. We shall show that there exist pairs of isothermic surfaces satisfying these equations.

Suppose that S is an isothermic surface. The parameters of its lines of curvature can be chosen so that

$$\sqrt{E} = \sqrt{G} = 1/\sqrt{\rho} = e^{\phi}, \tag{130}$$

where ϕ is a function thus defined. Now in the above equations

$$\sin\omega = \cos\omega = 1/\sqrt{2}, \qquad D = e^{2\phi}/\rho_1, \qquad D'' = e^{2\phi}/\rho_2,$$

where ρ_1 and ρ_2 are the principal radii of curvature of S. If we put

$$\alpha = \frac{a-b}{\sqrt{2}}, \qquad \beta = \frac{a+b}{\sqrt{2}}, \qquad c = -\frac{1}{m}, \tag{131}$$

and replace θ and λ by $m\theta$ and $m\lambda$, which evidently is consistent with equations (8), equations (115), (128), and (8) are equivalent to

$$\frac{\partial\alpha}{\partial u} = -m\lambda e^{\phi} + me^{-\phi}\theta - \frac{\partial\phi}{\partial v}\beta + \frac{we^{\phi}}{\rho_1}, \qquad \frac{\partial\alpha}{\partial v} = \beta\frac{\partial\phi}{\partial u},$$

$$
\begin{aligned}
&\frac{\partial\beta}{\partial u} = \alpha\frac{\partial\phi}{\partial v}, \qquad \frac{\partial\beta}{\partial v} = -m\lambda e^{\phi} - me^{-\phi}\theta - \frac{\partial\phi}{\partial u}\alpha + \frac{we^{\phi}}{\rho_2},\\
&\frac{\partial w}{\partial u} = -\alpha\frac{e^{\phi}}{\rho_1}, \qquad \frac{\partial w}{\partial v} = -\beta\frac{e^{\phi}}{\rho_2},\\
&\frac{\partial\theta}{\partial u} = e^{\phi}\alpha, \qquad \frac{\partial\theta}{\partial v} = e^{\phi}\beta, \qquad \frac{\partial\lambda}{\partial u} = -e^{-\phi}\alpha, \qquad \frac{\partial\lambda}{\partial v} = e^{-\phi}\beta.
\end{aligned} \tag{132}
$$

Furthermore equation (127) becomes

$$\alpha^2 + \beta^2 + w^2 + 2\lambda\theta m = 0. \tag{133}$$

It is readily shown that the first derivatives of the left hand member of equation (133) are zero, and consequently by choosing the initial values properly we can find solutions of equations (132) satisfying (133). It should be mentioned that all the conditions of integrability of (132) are easily shown to be satisfied.

If in the subsequent discussion X_1, Y_1, Z_1 and X_2, Y_2, Z_2 denote the direction-cosines of the tangents to the curves v = const. and u = const. respectively, it is necessary and sufficient to replace X_1 and X_2 in the foregoing formulas by $(X_1 + X_2)/\sqrt{2}$ and $(-X_1 + X_2)/\sqrt{2}$ respectively, as is evident from (104).

With these changes equations (113) and (114) become

$$x_1 - x = \frac{1}{\lambda m}(\alpha X_1 + \beta X_2 + wX) \tag{134}$$

and

$$
\begin{aligned}
&\frac{\partial x_1}{\partial u} = \frac{e^{-\phi}}{\lambda^2 m}[(\alpha^2 + \theta\lambda m)X_1 + \alpha\beta X_2 + \alpha wX],\\
&\frac{\partial x_1}{\partial v} = -\frac{e^{-\phi}}{\lambda^2 m}[\alpha\beta X_1 + (\beta^2 + \theta\lambda m)X_2 + \beta wX].
\end{aligned} \tag{135}
$$

From these equations we have for the fundamental coefficients E_1, F_1, G_1, of the surface S_1 the expressions

$$E_1 = G_1 = \frac{e^{-2\phi}\theta^2}{\lambda^2}, \qquad F_1 = 0, \tag{136}$$

which proves that S_1 also is isothermic.

For this case the direction-cosines X', Y', Z' of the normal to S_1, as given for the general case by (118), are of the form

$$X' = \frac{1}{\theta\lambda m}[(w^2 + \theta\lambda m)X + \alpha wX_1 + \beta wX_2]. \tag{137}$$

With the aid of this expression equation (134) may be written

$$x_1 + \frac{\theta}{w}X' = x + \frac{\theta}{w}X, \tag{138}$$

which shows that θ/w is the radius of the sphere tangent to S and S_1.

However, equations (132)–(138) are equivalent to those found by entirely different processes by Darboux* for the transformation of isothermic surfaces which he discovered and to which Bianchi has given the name *transformations* D_m.

In consequence of the foregoing results we have

THEOREM 14. *If two surfaces S and S_1 in the relation of a transformation K constitute the envelope of a two-parameter family of spheres, the surfaces are isothermic and S_1 is obtained from S by a transformation D_m.*

In accordance with (136) we define a function ϕ_1 by

$$e^{\phi_1} = \frac{e^{-\phi}\,\theta}{\lambda}. \tag{139}$$

If X_1', Y_1', Z_1' and X_2', Y_2', Z_2' denote the direction-cosines of the tangents to the parametric curves on S_1, it follows from (135) and (137) that

$$\begin{aligned} X_1' &= -\frac{1}{m\lambda\theta}[(\alpha^2 + \theta\lambda m)X_1 + \alpha\beta X_2 + \alpha w X], \\ X_2' &= \frac{1}{m\lambda\theta}[\alpha\beta X_1 + (\beta^2 + \theta\lambda m)X_2 + \beta w X]. \end{aligned} \tag{140}$$

On replacing X_1 and X_2 in (105) and (110) by $(X_1 + X_2)/\sqrt{2}$ and $(-X_1 + X_2)/\sqrt{2}$ respectively, we have the first derivatives of X, X_1, and X_2. With the aid of these formulas we obtain from (140)

$$\begin{aligned} \frac{\partial X'}{\partial u} &= \left[\frac{e^{\phi}}{\rho_1} + \frac{w}{\theta\lambda}(\lambda e^{\phi} - \theta e^{-\phi})\right]X_1', \\ \frac{\partial X'}{\partial v} &= -\left[\frac{e^{\phi}}{\rho_2} + \frac{w}{\theta\lambda}(\lambda e^{\phi} + \theta e^{-\phi})\right]X_2'. \end{aligned} \tag{141}$$

From these expressions and (135) it follows that the principal radii ρ_1' and ρ_2' of S_1 are given by

$$\begin{aligned} \frac{e^{\phi_1}}{\rho_1'} &= -\left[\frac{e^{\phi}}{\rho_1} + \frac{w}{\theta\lambda}(\lambda e^{\phi} - \theta e^{-\phi})\right], \\ \frac{e^{\phi_1}}{\rho_2'} &= \left[\frac{e^{\phi}}{\rho_2} + \frac{w}{\theta\lambda}(\lambda e^{\phi} + \theta e^{-\phi})\right]. \end{aligned} \tag{142}$$

* Loc. cit., p. 502.

10. Theorem of permutability of transformations D_m

In this section we apply the results of § 3 to an immediate proof of the theorem of permutability of transformations D_m which Bianchi* established at length by direct processes. We are thus able to appreciate the underlying reason for the existence of this theorem.

In the first place we seek solutions α_1', β_1', w_1', θ_1', λ_1' of equations for S_1 analogous to (132) and (133), it being understood that two sets of solutions of the latter equations are known, which we denote by α_i, β_i, w_i, θ_i, λ_i for $i = 1, 2$. The functions θ_1' and λ_1' are given by equations (27) and (36). Hence if α_1' and β_1' are to satisfy

$$\frac{\partial\theta_1'}{\partial u} = e^{\phi_1}\alpha_1' = -\frac{e^{-\phi}\theta_1}{\lambda_1}\alpha_1', \qquad \frac{\partial\theta_1'}{\partial v} = -\frac{e^{-\phi}\theta_1}{\lambda_1}\beta_1', \tag{143}$$

we must have

$$\begin{gathered}\alpha_1' = -\frac{\alpha_1}{\theta_1}(\theta_1' + \theta_2) + \alpha_2, \qquad \beta_1' = \frac{\beta_1}{\theta_1}(\theta_1' + \theta_2) - \beta_2, \\ \lambda_1' = \frac{\lambda_1}{\theta_1}(\theta_1' + \theta_2) - \lambda_2.\end{gathered} \tag{144}$$

These values of α_1' and β_1' satisfy identically the equations

$$\frac{\partial\alpha_1'}{\partial v} = \beta_1'\frac{\partial\phi_1}{\partial u}, \qquad \frac{\partial\beta_1'}{\partial u} = \alpha_1'\frac{\partial\phi_1}{\partial v}, \tag{145}$$

and when substituted in

$$\frac{\partial\alpha_1'}{\partial u} = -m_1'\lambda_1' e^{\phi_1} + m_1' e^{-\phi_1}\theta_1' - \frac{\partial\phi_1}{\partial v}\beta_1' + w_1'\frac{e^{\phi_1}}{\rho_1'},$$

$$\frac{\partial\beta_1'}{\partial v} = -m_1'\lambda_1' e^{\phi_1} - m_1' e^{-\phi_1}\theta_1' - \frac{\partial\phi_1}{\partial u}\alpha_1' + w_1'\frac{e^{\phi_1}}{\rho_2'},$$

and use is made of the foregoing equations, we get

$$A\frac{e^{\phi_1}}{\rho_1'} + (\theta_1 e^{-\phi} - \lambda_1 e^{\phi})B + \lambda_1 e^{\phi}\left(m_2\frac{\lambda_2}{\lambda_1} + m_1'\frac{\theta_2}{\theta_1}\right) - \theta_1 e^{-\phi}\left(m_1'\frac{\lambda_2}{\lambda_1} + m_2\frac{\theta_2}{\theta_1}\right) = 0,$$

$$A\frac{e^{\phi_1}}{\rho_2'} + (\theta_1 e^{-\phi} + \lambda_1 e^{\phi})B - \lambda_1 e^{\phi}\left(m_2\frac{\lambda_2}{\lambda_1} + m_1'\frac{\theta_2}{\theta_1}\right) - \theta_1 e^{-\phi}\left(m_1'\frac{\lambda_2}{\lambda_1} + m_2\frac{\theta_2}{\theta_1}\right) = 0,$$

* Loc. cit., pp. 109–125.

where

$$A = w_1' - \frac{w_1}{\theta_1}(\theta_1' + \theta_2) + w_2,$$

$$B = \frac{1}{\theta_1}\left[(\theta_1' + \theta_2)(m_1' - m_1) - \frac{1}{\lambda_1}(\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2)\right]. \tag{146}$$

If we put $m_1' = m_2$, these equations become

$$A\frac{e^{\phi_1}}{\rho_1'} + \frac{C}{\theta_1}(\theta_1 e^{-\phi} - \lambda_1 e^{\phi}) = 0, \qquad A\frac{e^{\phi_1}}{\rho_2'} + \frac{C}{\theta_1}(\theta_1 e^{-\phi} + \lambda_1 e^{\phi}) = 0, \tag{147}$$

where

$$C = (\theta_1' + \theta_2)(m_2 - m_1) - \frac{1}{\lambda_1}(\alpha_1\alpha_2 + \beta_1\beta_2 + w_1 w_2) - \frac{m_2}{\lambda_1}(\lambda_2\theta_1 + \lambda_1\theta_2). \tag{148}$$

Equations (147) are satisfied when $A = C = 0$. From these equations and (144), we have

$$\begin{aligned} -\alpha_1' &= \kappa\left[\frac{\Phi_1\alpha_1}{\theta_1\lambda_1} + (m_1 - m_2)\alpha_2\right], \\ \beta_1' &= \kappa\left[\frac{\Phi_1\beta_1}{\theta_1\lambda_1} + (m_1 - m_2)\beta_2\right], \\ w_1' &= \kappa\left[\frac{\Phi_1 w_1}{\theta_1\lambda_1} + (m_1 - m_2)w_2\right], \\ \lambda_1' &= \kappa\left[\frac{\Phi_1}{\theta_1} + (m_1 - m_2)\lambda_2\right], \\ \theta_1' &= \kappa\left[\frac{\Phi_1}{\lambda_1} + (m_1 - m_2)\theta_2\right], \qquad \kappa = \frac{1}{m_2 - m_1}, \end{aligned} \tag{149}$$

where we have put

$$\Phi_1 = \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 + m_2(\lambda_2\theta_1 + \lambda_1\theta_2). \tag{150}$$

It is readily shown that these values of θ_1' and w_1' satisfy equations (143) and also

$$\frac{\partial w_1'}{\partial u} = -\alpha_1'\frac{e^{\phi_1}}{\rho_1'}, \qquad \frac{\partial w_1'}{\partial v} = -\beta_1'\frac{e^{\phi_1}}{\rho_2'},$$

and consequently the functions (149) satisfy all the conditions necessary to the problem. Hence we have established with Bianchi

THEOREM 15. *If S is an isothermic surface, and two isothermic surfaces S_1 and S_2 are obtained from S by transformations D_{m1} and D_{m2}, there exists a*

transformation D'_{m2} of S_1 into a surface S' which can be obtained from S_2 also by a transformation D'_{m1}; moreover, S' can be found without quadratures.

When S is isothermic, the associate surface defined by equations (11) is the Christoffel transform. Hence, as an immediate consequence of Theorem 5, we have that transformations D_m are commutative with transformations of Christoffel, a result which Bianchi established by direct calculation.

11. Congruences (G) with more than one pair of surfaces in the relation of a transformation K

In this section we consider the consistency of equations (13) and (18). We write the former in the form of the second of equations (15) and express the condition of integrability. This gives

$$\frac{\partial M}{\partial v} = \frac{k}{N^2} k_{-1} + M \frac{\partial \log k}{\partial v} + 2M^2 - \frac{3Mk}{N}, \tag{151}$$

where k_{-1}, given by

$$k_{-1} = 2k - h - \frac{\partial^2 \log k}{\partial u \, \partial v}, \tag{152}$$

is an invariant of the point equation of the second focal surface of tangents to the curves $u = \text{const.}$ on S_0.

The condition of integrability of (151) and of the first of (15) can be reduced to

$$N^2 h \frac{\partial}{\partial v} \log \frac{h}{k} + \left(k_{-1} \frac{\partial k}{\partial u} - k \frac{\partial k_{-1}}{\partial u} \right) + 3N(MN - k)(k_{-1} - h) = 0. \tag{153}$$

Hence the necessary and sufficient condition that (151) and (15) be consistent identically is that

$$k_{-1} = h = (c + 1)k, \tag{154}$$

where c denotes an arbitrary constant. In this case it follows from (152) and (154) that

$$\frac{\partial^2 \log k}{\partial u \, \partial v} + 2ck = 0. \tag{155}$$

When $c = 0$, we have

$$h = k = UV, \tag{156}$$

where U and V are arbitrary functions of u and v respectively. When $c \neq 0$, the general integral of (155) is

$$k = \frac{U' V'}{c(1 + UV)^2}, \qquad h = (c + 1)k, \tag{157}$$

U and V being arbitrary functions of u and v respectively and a prime indicating differentiation with respect to the argument.

When $c = -1$, $h = 0$ and it is readily shown that in this case S and S_1 coincide. Consequently we exclude this value.

If we take the expressions (156) or (157) for h and k, and determine a and b in accordance with (14′) two arbitrary functions are introduced. Then each of the functions M and N as determined by equations (15), (18), and (151) involves an arbitrary constant, and σ and σ_1 are found by quadratures. In view of these results we have

THEOREM 16. *Whenever the point equation of a surface S_0 has invariants of the form* (156) *or* (157), *there exist on the tangents to the curves $v = \text{const.}$ of S_0 an infinity of pairs of points which generate pairs of surfaces in the relations of transformations K; the same is true of the tangents to the curves $u = \text{const.}$ of S_0, if $h = k = UV$.*

Returning to the case when equation (153) is not satisfied identically, we differentiate this equation respectively with respect to u and v, and replace the derivatives of M and N by their expressions from (15), (18), and (151). By means of (153) the two resulting equations are reducible to the forms

$$PN^2 + Q = 0, \qquad RN^2 + S = 0, \tag{158}$$

where P, Q, R, S are determinate functions involving h, k, k_{-1} and their derivatives. When we express the conditions that these equations be consistent, and that the expressions for N thus given and of M from (153) shall satisfy equations (15), (18), (151), we obtain the equations for h and k which determine the surfaces S_0. We are not interested in these equations but in the fact that from the form of equations (158) and (16) it follows that there is only one pair of surfaces S and S_1 associated with the given congruence. Hence we have

THEOREM 17. *If a congruence (G) possesses more than one pair of points which generate surfaces in the relation of a transformation K, it possesses an infinity of such pairs.*

PRINCETON UNIVERSITY,
April 14, 1914.

PROOF OF THE FINITENESS OF THE MODULAR COVARIANTS OF A SYSTEM OF BINARY FORMS AND COGREDIENT POINTS*

BY

FORBES BAGLEY WILEY

INTRODUCTION

1. **Relation to the literature.** The question of the finiteness of the modular covariants of a system of forms and cogredient points is one that arises naturally at this time in view of the fact that the modular covariants of a system of forms and no cogredient points have been shown recently to be finite,† where the term finite has the customary meaning attached to it in covariant theory, and still more recently a fundamental system of modular covariants for two cogredient binary points and no forms has been found.‡

The present paper is limited to the binary case, although the writer has made considerable advance in the case of ternary forms and one cogredient point.

2. **Definitions.** Let $f_1, \cdots, f_l$ be any system of forms (homogeneous polynomials) in the arbitrary variables x_{01}, x_{02} with undetermined integral coefficients taken modulo p, where p is any prime. Let $c_1, c_2, \cdots$ denote the coefficients arranged in any order. Under the transformation

$$T_0: \qquad x_{0i} = \sum_{j=1}^{2} t_{ij} x'_{0j} \pmod{p} \qquad (i = 1, 2)$$

with integral coefficients, let f_i become the form f'_i and let $c'_1, c'_2, \cdots$ denote the coefficients of $f'_1, \cdots, f'_l$ corresponding in position to $c_1, c_2, \cdots$, respectively. Let (x_{k1}, x_{k2}), $k = 1, \cdots, n$, be transformed cogrediently with x_{01}, x_{02}, that is by the transformation

$$T_k: \qquad x_{ki} = \sum_{j=1}^{2} t_{ij} x'_{kj} \pmod{p} \quad (i = 1, 2; k = 1, \cdots, n).$$

A rational integral function

$$K(c_1, c_2, \cdots; x_{01}, x_{11}, \cdots, x_{n1}; x_{02}, x_{12}, \cdots, x_{n2})$$

* Presented to the Society (Providence), September 8, 1914.

† Dickson, these Transactions, vol. 14 (1913), pp. 299–305.

‡ Krathwohl, American Journal of Mathematics, October, 1914.

with integral coefficients taken modulo p is called a modular covariant of the points (x_{k1}, x_{k2}), $k = 1, \cdots, n$, and the forms $f_1, \cdots, f_l$ if, for every transformation T_k $(k = 0, 1, \cdots, n)$,

$$K(c_1', c_2', \cdots; x_{01}', x_{11}', \cdots, x_{n1}'; x_{02}', x_{12}', \cdots, x_{n2}')$$

$$\equiv |t_{ij}|^{\mu} K(c_1, c_2, \cdots; x_{01}, x_{11}, \cdots, x_{n1}; x_{02}, x_{12}, \cdots, x_{n2}) \pmod{p}$$

holds identically in $c_1, c_2, \cdots; x_{01}', \cdots, x_{n1}'; x_{02}', \cdots, x_{n2}'$ after $x_{01}, \cdots, x_{n1}$, $x_{02}, \cdots, x_{n2}$ are eliminated by means of the congruences $T_k (k = 0, \cdots, n)$ and $c_1', c_2', \cdots$ are replaced by their expressions in terms of $c_1, c_2, \cdots$. The exponent μ is called the index of K.

In this paper we establish the

THEOREM. *The set of all modular covariants of any system of forms in x_{01}, x_{02} and the cogredient points (x_{k1}, x_{k2}), $k = 1, \cdots, n$, is finite in the sense that they are all rational integral functions, with integral coefficients taken modulo p, of a finite number of covariants of the set.*

Professor Dickson has obtained the special case of this theorem where there is a system of forms but no cogredient points.* We extend his method of proof to secure the proof of the theorem as above stated.

3. **Method of proof.** In bare outline the proof of the theorem consists of two parts. First it is shown that for any modular covariant K_1 of the forms and the points there exists a modular covariant K_1' that has the same leader (defined in § 6) and the same index as K_1 has and that is a polynomial in covariants from a definite finite set S of modular covariants of the forms and the points. By taking the difference of K_1 and K_1' we obtain a new covariant K_2 for which we can build up K_2' by the same method as that used on K_1 and by the use of the same set S. The second part of the proof consists in showing that the repetition of this process a finite number of times on any given covariant K_1 leads to a covariant K_m where K_m is such that $K_m - K_m' = 0$. Thus it follows that $K_1 = \Sigma_{i=1}^{i=m} K_i'$ and therefore is expressible as a polynomial in covariants from the set S.

To carry out this method in detail we establish in § 4 a lemma which is an extension of the lemma† used by Professor Dickson in the case he considered. In § 5 we prove the theorem for n not greater than 1 and in § 6 we give the proof where n is any finite positive integer, including the value $n = 0$.

* Dickson, loc. cit. Professor Dickson secures his theorem for forms in any finite number of variables.

† Dickson, American Journal of Mathematics, vol. 35 (1913), pp. 414–415.

Finiteness of the Modular Covariants

4. **Lemma.** *Any set S of functions of the type*

$$F = x_0^{e_0} x_1^{e_1} \cdots x_n^{e_n} \quad (e\text{'s integers} \geqq 0;\ e_0 \geqq e_1, \cdots, e_0 \geqq e_n) \tag{1}$$

contains a finite number of functions $F_1, \cdots, F_k$ such that each function F of the set S can be expressed as a product $F_i f$ where f is of the form (1), *with the same property of the exponents, but is not necessarily in the set S.*

For the moment designate the possible relations

$$e_i - e_{i+1} \geqq 0, \qquad e_i - e_{i+1} < 0 \qquad (i = 1, \cdots, n)$$

between the exponents of F as decreases and increases, respectively. Since the e's of F occur in a definite order, indicated in (1), the order in which the increases and decreases of a given F occur is unique. The functions F of S can be classified into sets S_j $(j = 1, \cdots, t; t \leqq n!)$ such that functions from no two sets will have their increases and decreases occurring in the same order, while the increases and decreases of functions in the same set occur in the same order.

Let S_k be one of the sets S_j, say the one in which there are no increases. It is evident that the functions of any set S_j can be written in the form of the functions of S_k by a substitution which amounts to a rearrangement of the x's. Write the functions $F^{(k)}$ of S_k in the form

$$\prod_{j=0}^{n}\left[\prod_{i=0}^{j} x_i\right]^{e_j - e_{j+1}} \qquad (e_{n+1} = 0).$$

Make the substitutions

$$\prod_{i=0}^{j} x_i = y_j, \qquad e_j - e_{j+1} = \epsilon_j \qquad (j = 1, \cdots, n) \tag{2}$$

on the variables and the exponents of $F^{(k)}$, respectively. We thus have a set S_k of functions

$$\phi^{(k)} = \prod_{j=0}^{n} y_j^{\epsilon_j} \qquad (\epsilon\text{'s integers} \geqq 0).$$

There exists* a finite number of functions $\phi_i^{(k)}$ $(i = 1, \cdots, h)$ belonging to the set S_k such that any function $\phi^{(k)}$ of the set S_k may be written

$$\phi^{(k)} = \phi_i^{(k)} f,$$

where

$$f = \prod_{j=0}^{n} y_j^{a_j} \qquad (a\text{'s integers} \geqq 0) \tag{3}$$

and i has some one of the values $1, \cdots, h$. Transforming (3) by the inverse

* See reference last cited; also Gordan,* Journal de Mathématiques, ser. 5, vol. 6 (1900), pp. 141–149.

of (2_1), we have

$$f = \prod_{j=0}^{n} x_j^{\sum_{i=j}^{n} a_i}.$$

The lemma thus holds for the set S_k and similarly for each one of the t sets S_j; hence it holds for the set S.

5. **One cogredient point.** We make use of the known* rational integral modular covariants

$$L_i = x_{i1}^p x_{i2} - x_{i1} x_{i2}^p \qquad (i = 0, 1),$$

$$Q_i = \frac{x_{i1}^{p^2} x_{i2} - x_{i1} x_{i1}^{p^2}}{L_i} = x_{i1}^{p^2-p} + x_{i2}(\,) \qquad (i = 0, 1),$$

$$M = x_{01} x_{12} - x_{02} x_{11}.$$

The covariants L_i and M have the index -1, while the Q_i are absolute.†

As any covariant is the sum of covariants that are homogeneous in each pair of variables separately, it is sufficient for us to consider covariants of the type $K(x_{01}, x_{02}, x_{11}, x_{12})$ of total degree ω_0 in x_{01}, x_{02} and of total degree ω_1 in x_{11}, x_{12}. If one of the ω's is zero, we have the case which Professor Dickson considered.

We shall call such a covariant K a regular covariant of the forms and the cogredient point if it has neither of the x_{i2} $(i = 1, 2)$ as a factor, while we shall call it irregular if it has either one of these variables as a factor or both as factors. Any irregular covariant is the product of $L_0^s L_1^t$ $(s, t = 0, 1, \cdots;$ s and t not both zero) by a regular covariant, since‡ any covariant that has the factor x_{i2} has the factor L_i. We need, therefore, to consider only regular covariants.

Let K be such a covariant and let it be of total degree ω_i in the two variables x_{i1}, x_{i2} $(i = 0, 1)$. We write K in the form

$$K = x_{i1}^{\omega_i} \sum_{j=0}^{\omega_k} S_j x_{k1}^{\omega_k - j} x_{k2}^{j} + x_{i2}(\,) \qquad (\omega_i \geqq \omega_k;\ i, k = 0, 1;\ i \neq k), \tag{4}$$

where at least one S_j is not zero. We rewrite (4) in the notation

$$K = S x_{i1}^{A} x_{k1}^{B} x_{k2}^{C} + \cdots \qquad (S = S_0,\ A = \omega_i,\ B + C = \omega_k), \tag{5}$$

where B is the highest exponent which x_{k1} carries in a term containing x_{i1}^A. Denote the set of all regular covariants (5) with a fixed leader S and

$$A \equiv a, \qquad B \equiv b, \qquad C \equiv c \pmod{P} \qquad (P = p^2 - p),$$

* Dickson, *The Madison Colloquium Lectures on Mathematics*, pp. 35, 37. M is an algebraic covariant.

† Of index $\mu = 0$.

‡ Dickson, *The Madison Colloquium Lectures*, p. 35.

where each of the numbers a, b, c is at least zero and is less than P, by $[S, a, b, c]_i$. The number of such sets is finite since a, b, c are limited in value and the number of seminvariant leaders S is finite. These leaders are finite because each S is a polynomial in the coefficients c_i of the forms and thus all exponents can be reduced to $p-1$ or less by use of Fermat's theorem. All covariants belonging to the same set S have the same index μ. By means of the Lemma (§ 4), we may select from the set $[S, a, b, c]$ a finite number t of covariants

$$K_h = Sx_{i1}^{A_h} x_{k1}^{B_h} x_{k2}^{C_h} + \cdots + x_{i2}(\,) \qquad (h = 1, \cdots, t), \tag{6}$$

such that for any covariant (5) of the set $[S, a, b, c]$ it is true that

$$A \geqq A_h, \qquad B \geqq B_h, \qquad C \geqq C_h, \qquad A - A_h \geqq C - C_h$$

$$(\text{for some value} \leqq t \text{ of } h).$$

But A and A_h (and likewise for B and B_h, C and C_h) are congruent to a modulo P; thus any covariant of the set $[S, a, b, c]$ has the form

$$K = Sx_{i1}^{A_h+uP} x_{k1}^{B_h+vP} x_{k2}^{C_h+wP} + \cdots \qquad (u \geqq w),$$

where u, v, w are positive integers or zero and h is some one of the integers $1, \cdots, t$.

From the absolute covariants Q_i, Q_k, M^P, and the covariant K_h, we construct a covariant of the same index as K and having the same initial term as K, and see that the difference

$$K' = K - K_h Q_i^{u-w} Q_k^{v} M^{wP} + \cdots$$

will be a covariant either irregular or with the leader

$$S' x_{i1}^{A} x_{k1}^{B'} x_{k2}^{C'} \qquad (A \geqq B' + C',\ B' < B,\ C' > C).$$

If K' is an irregular covariant, take out the factors L_i that occur and call the quotient K''. If K' is regular, call it K''. Hence

$$K = \psi_1(L_1, L_2, Q_1, Q_2, M, K_h, K''),$$

where ψ_1 is a polynomial in its arguments.

Arrange K'' in the form (6) and repeat for it the reduction process used on the regular covariant K. Whenever the exponent B in one of the regular covariants, say $K^{(e)}$, is zero, the next step in the process gives an irregular covariant with the factor x_{i2}, thus allowing a reduction of the degree in x_{i1} and x_{i2} by the removal of the factor L_i. If we let K_h $(h = 1, \cdots, \tau)$ serve for all sets $[S, a, b, c]$ as K_h $(h = 1, \cdots, t)$ served for the particular one, we see that every covariant K may be expressed as

$$K = \psi_m(K_1, \cdots, K_\tau, L_1, L_2, Q_1, Q_2, M),$$

where ψ_m is a polynomial in its arguments and is linear in the $K_1, \cdots, K_\tau$.

6. **Case of n cogredient points.** We use the L_i and the Q_i of § 5 with $i = 1, \cdots, n$. In addition to these we use the known modular covariants

$$M_{ik,n} = x_{i1}\, x_{k2}^{p^n} - x_{i2}\, x_{k1}^{p^n} \quad (i, k = 0, \cdots, n;\ i \neq k).$$

As in § 5 so here also we need consider only those covariants of the forms $f_1, \cdots, f_l$ and the cogredient points (x_{k1}, x_{k2}), $k = 1, \cdots, n$, which are homogeneous in each pair of variables separately.

We shall call such a covariant a regular covariant if it has none of the x_{i2} $(i = 0, \cdots, n)$ as factors, while we shall call it irregular if it has one or more such factors. Since any irregular covariant is the product of

$$\prod_{i=0}^{n} L_i^{s_i} \qquad (s_i = 0, 1, \cdots; \text{ at least one } s_i \neq 0)$$

and a regular covariant, we need to consider only regular covariants.

Let K be such a covariant and let ω_i be the order of K in x_{i1}, x_{i2}. Each x_{i1} will occur in at least one term of K with an exponent equal to the order ω_i. There exists some ω_i, say ω_0 for convenience in notation,* such that $\omega_0 \geqq \omega_j$ $(j = 1; \cdots, n)$. Write K with those terms leading that contain the factor $x_0^{\omega_0}$, the order of these terms being such that the sums of the exponents of the factors x_{i1} in them occur in the order of the decreasing natural numbers. The order among themselves of those terms for which such sums are equal is arbitrary. Denote the first term of K so written as the leader. By the help of zero exponents where necessary, write the leader of K so that every variable except x_{02} appears in it notationally as a factor. Arrange these factors in the ascending order of their subscripts so that we have

$$K = S \prod_{i=0}^{n} x_{i1}^{A_i} \prod_{j=1}^{n} x_{j2}^{B_j} + \cdots \qquad (A_0 = \omega_0,\ A_i + B_i = \omega_i,\ \omega_0 \geqq \omega_i). \tag{7}$$

Denote by

$$[S, a_0, a_1, \cdots, a_n; b_1, b_2, \cdots, b_n] \tag{8}$$

the set of all covariants (7) with a given seminvariant leader S and with

$$A_i \equiv a_i, \qquad B_i \equiv b_i \pmod{X}; \qquad X = p^{n+1}(p-1),$$

where each a_i and each b_i is at least zero and less than X. The number of sets (8) is finite. All covariants (7) belonging to the same set (8) have the same index μ. By the Lemma (§ 4), there exists a finite number of covariants

$$K_h = S \prod_{i=0}^{n} x_{i1}^{A_{ih}} \prod_{j=1}^{n} x_{j2}^{B_{jh}} + \cdots \qquad (h = 1, \cdots, t) \tag{9}$$

belonging to the set (8) such that the leader of any covariant (7) that belongs

* It is only slightly more cumbersome to carry the argument through for ω_g.

to set (8) can be expressed as the product of the leader of some covariant of (9) by f, where f is of the form (7), with the same property of the exponents, but has no factor S. Thus we may write

$$K = S\prod_{i=0}^{n} x_{i1}^{A_{ih}+q_iX}\prod_{j=1}^{n} x_{j2}^{B_{jh}+q'_jX} + \cdots. \tag{10}$$

Therefore

$$f = \left(\prod_{i=0}^{n} x_{i1}^{q_i}\prod_{j=1}^{n} x_{j2}^{q'_j}\right)^X$$
$$(q_0 \geqq q_1, \cdots, q_0 \geqq q_n;\ q_0 \geqq q'_1, \cdots, q_0 \geqq q'_n).$$

By the aid of zero exponents where necessary, let each variable except x_{02} appear notationally as a factor in f. Arrange the factors x_{i2} of f in the order of decreasing exponents. The order among themselves of those factors x_{i2} whose exponents are equal is arbitrary. Arrange the x_{i1} factors in the same order as their corresponding x_{i2}'s. Write f thus arranged in the notation

$$f = \left(x_{01}^{q_0}\prod_{i=1}^{n} x_{h_i1}^{q_{h_i}}\prod_{j=1}^{n} x_{h_j2}^{q'_{h_j}}\right)^X.$$

This may be written

$$f = \Bigg[x_{01}^{q_0-q'_{h_1}}\prod_{i=1}^{n} x_{h_i1}^{q_{h_i}}(x_{01}\,x_{h_12})^{q'_{h_1}-q'_{h_2}}(x_{01}\,x_{h_12}\,x_{h_22})^{q'_{h_2}-q'_{h_3}} \cdots (x_{01}\cdots x_{h_{n-1}2})^{q_{h_{n-1}}-q'_{h_n}}(x_{01}\cdots x_{h_n2})^{q'_{h_n}}\Bigg]^X. \tag{11}$$

We now build up the absolute covariants

$$P_r = Q_0^{p^n-r}\prod_{k=h_1}^{h_r} M_{0k,n}^{p^2-p} + \cdots \qquad (1 \leqq r \leqq n),$$

which we may write in the form

$$P_r = \left[x_{01}\prod_{k=h_1}^{h_r} x_{h_k2}\right]^X + \cdots.$$

From the absolute covariants Q_0, Q_{h_i}, P_r, and the covariant K_h, which is one of the covariants (9), we construct a covariant of the same index as K and having the same initial term as K so that it follows that the difference

$$K' = K - K_h\,Q_0^{(q_0-q'_1)p^n}\prod_{i=1}^{n} Q_{h_i}^{q_{h_i}-p^n}\prod_{r=1}^{n} P_r^{q'_{h_j}-q'_{h_{j+1}}} \qquad (q'_{h_{n+1}}=0)$$

will be a covariant either irregular or with a leader that can be written in the form (10). If K' is irregular, take out the factors L_i that occur and call the quotient K''. If K' is regular, call it K''. Thus we have

$$K = \psi(L_1, \cdots, L_n, Q_1, \cdots, Q_n, K_h, P_1, \cdots, P_l)$$

where h is one of the values $1, \cdots, t$ and ψ is a polynomial in its arguments. Arrange K'' in the form (10) and subject it to the reduction process used on the regular covariant K, thus obtaining a regular covariant $K^{(IV)}$. After a finite number of repetitions of this process, we obtain either an irregular covariant or a regular one, say $K^{(q)}$, for which the sum of the exponents of the factors x_{i1} $(i = 1, \cdots, n)$ in its leader is less than the like sum for the leader of K. After repeating the process a finite number of times from this point, we obtain either an irregular covariant or one, say $K^{(l)}$, for which the sum of the exponents of the x_{i1} $(i = 1, \cdots, n)$ factors is zero. The next repetition of the reduction process reduces the exponent of x_{01} and gives an irregular covariant with the factor L_0 which, when removed, still further reduces the degree of x_{01}.

If we let K_h $(h = 1, \cdots, \tau)$ serve for all sets (8) as K_h $(h = 1, \cdots, t)$ served for one of them, we see that any covariant of the binary forms $f_1, \cdots, f_t$ and the n cogredient points is a polynomial in $K_1, \cdots, K_\tau$; $L_1, \cdots, L_n$; $Q_1, \cdots, Q_n$; $P_1, \cdots, P_n$, linear in $K_1, \cdots, K_\tau$.

The covariants P_r are polynomials in Q_0 and $M_{0k,\,n}$. It is of interest to note that by means of the recursion formula

$$M_{0k,\,n} = Q_k^{p^{n-2}} M_{0k,\,n-1} - L_k^{(p-1)p^{n-2}} M_{0k,\,n-2} \qquad (n > 1),$$

we are able to express any $M_{0k,\,n}$ as a rational integral function of $M_{0k,\,1}$, $M_{0k,\,0}$, Q_k, and L_k, linear in the $M_{0k,\,1}$, $M_{0k,\,0}$. It follows that any covariant K of the binary forms and the n points may be written as a polynomial in $K_1, \cdots, K_\tau$; $L_1, \cdots, L_n$; $Q_1, \cdots, Q_n$; $M_{01,\,1}, \cdots, M_{0n,\,1}$; $M_{01,\,0}, \cdots, M_{0n,\,0}$, linear in the $K_1, \cdots, K_\tau$.

ON THE DEGREE OF CONVERGENCE OF STURM-LIOUVILLE SERIES*

BY

DUNHAM JACKSON

After the degree of convergence of Fourier's series has been studied to a certain extent, as has been done by various authors, including the present writer, it is natural to inquire whether the results obtained are capable of extension to other series of characteristic solutions of homogeneous linear differential equations. A theorem in this connection has already been published by Tamarkine.† It is the purpose of the present paper to develop a greater variety of facts relating to a much less general differential equation than that which Tamarkine treats. In discussing expansion problems connected with an equation of the second order, Kneser,‡ following Liouville,§ establishes relations from which it is at once apparent that the theorem of Lebesgue,|| which states that the remainder after n terms in the Fourier's series for a function satisfying a Lipschitz condition does not exceed a constant multiple of $(\log n)/n$, applies equally well to the development of such a function in Sturm-Liouville series of the form under consideration. It is to be shown how these relations can be refined so as to make it possible to carry over other theorems of the same nature from the Fourier's series to the more general one. The theorems susceptible of this generalization include that of Picard* based on the hypothesis that the function developed has a kth derivative of limited variation, and one proved by the author† for a function having

* Presented to the Society, December 30, 1913.

† Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), pp. 345–382; see especially p. 368. In connection with this paper, though not with reference to the particular point mentioned here, see also, in the same journal, Birkhoff, vol. 36 (1913), pp. 115–126, and Tamarkine, vol. 37 (1914), pp. 376–378.

‡ Mathematische Annalen, vol. 58 (1904), pp. 81–147; see p. 127.

§ Journal de mathématiques pures et appliquées, vol. 2 (1837), pp. 418–436.

|| Bulletin de la société mathématique de France, vol. 38 (1910), pp. 184–210; p. 201.

* *Traité d'Analyse*, vol. 1, chapter IX, § 12. Picard is not directly concerned with the degree of convergence of the series, but nothing more is needed in this case than the formulas which he gives for the order of magnitude of the coefficients. The hypothesis concerning the function also is not exactly that stated here, but is essentially the same for the purposes of the proof. When the Lipschitz condition is used a knowledge of the order of magnitude of the coefficients is not enough; see Lebesgue, loc. cit., pp. 192–195.

† These Transactions, vol. 13 (1912), pp. 491–515, Theorem V. This paper will be referred to as A.

a $(k-1)$th derivative that satisfies a Lipschitz condition. The case of a function having a finite second derivative, which was treated by Liouville,* is included in that of functions having a first derivative of limited variation. The method employed is intimately related with that by which Horn† obtains extended asymptotic expressions for solutions of the differential equation.

For the proofs relating to the higher degrees of convergence, additional hypotheses, beyond that of mere continuity, are made concerning the functions which appear as coefficients in the differential equation, and it is assumed that the function developed vanishes with a sufficient number of its derivatives at the ends of the interval. The latter requirement may seem unduly restrictive, but an examination of the facts in the simplest case, that of the cosine-series,‡ shows that a limitation of this nature is required for the truth of the conclusions, and is not due merely to inadequacy of the method of treatment.

In the last part of the paper attention is given to the problem of representing a function that has a $(k-1)$th derivative satisfying a Lipschitz condition by means of a linear combination of a finite number of characteristic functions, with a higher degree of approximation than that afforded by the sum of the corresponding terms in the Sturm-Liouville series itself. Such a representation is obtained by summing the series by a method which the author had previously applied to Fourier's series.

1. Preliminary Statements

The differential equation with which we shall deal is the following:

$$\frac{d^2U}{dx^2} + [\rho^2 - \lambda(x)]\,U = 0. \tag{1}$$

Here $\lambda(x)$ is a function which is assumed at the outset to be continuous, and will be subjected to further restrictions as occasion demands, but only when such restrictions are explicitly mentioned. The parameter ρ^2 is not restricted to positive nor even to real values. The familiar transformation§ which reduces a more general equation to this form gives so directly the interpretation of our results with reference to the general equation that it is unnecessary to dwell upon the latter. The boundary conditions are

$$\begin{aligned} U'(0) - h'\,U(0) &= 0, \\ U'(\pi) + H'\,U(\pi) &= 0, \end{aligned} \tag{2}$$

*Journal de Mathématiques pures et appliquées, vol. 2 (1837) pp. 16–35; see especially the sentence beginning at the foot of p. 18. See also Kneser, loc. cit., pp. 121–123.

†Mathematische Annalen, vol. 52 (1899), pp. 271–292.

‡It is the cosine-series, rather than the complete Fourier's series, which is the true prototype of the expansions considered in this article.

§See, e. g., Kneser, loc. cit., pp. 116–117.

the interval over which the variable x is to range being taken for convenience as that from 0 to π. The numbers h' and H' are real constants, not restricted as to sign; the notation is the classical one. The limiting cases obtained by letting one or both of these constants become infinite, that is, by making the solution U vanish at one or both ends of the interval, while requiring a discussion differing in details from the one that is to be given, offer nothing essentially new that is not provided for in the paper of Kneser already cited. and will not be treated separately here.

The following facts concerning the characteristic numbers and solutions of the system consisting of the equation (1) and the boundary conditions (2) will be assumed as well known.

There are infinitely many real* values of ρ^2 for which the system has a real solution not identically zero. Only a finite number of these values can be negative, and they have no cluster-point in the finite plane. If ρ_n represents the positive square root of the nth of them in algebraic order of magnitude, when n is large enough so that the corresponding value of ρ^2 is positive, then

$$\rho_n = n + \epsilon_n, \tag{3}$$

where†

$$\epsilon_n = O\left(\frac{1}{n}\right). \tag{4}$$

Two linearly independent solutions can not correspond to the same value of ρ^2. If the arbitrary constant factor in the nth characteristic solution $U_n(x)$ is suitably determined, the solution satisfies the integral equation

$$U_n(x) = \cos \rho_n x + \frac{h' \sin \rho_n x}{\rho_n} + \frac{1}{\rho_n}\int_0^x \lambda(t)\, U_n(t) \sin \rho_n (x - t)\, dt, \tag{5}$$

when the corresponding value of ρ^2 is positive. The maximum value of $U_n(x)$ in the interval $(0, \pi)$ remains finite as n becomes infinite,‡ so that the third as well as the second term on the right-hand side is very small when n is very large.

We shall be interested in the degree of convergence of the series

$$\sum_{n=0}^{\infty} \alpha_n U_n(x), \tag{6}$$

where

$$\alpha_n = \frac{\int_0^\pi \phi(x)\, U_n(x)\, dx}{\int_0^\pi [U_n(x)]^2\, dx}, \tag{7}$$

* The fact that there are no complex characteristic values will not be used explicitly.

† This notation means that the absolute value of ϵ_n does not exceed a constant multiple of $1/n$.

‡ This is readily deduced from (5) itself.

and $\phi(x)$ is a continuous function on which further restrictions will be imposed later. It will be assumed as known* that if this series converges uniformly, its value must necessarily be $\phi(x)$.

With this review of facts that are presupposed,† we go on to the detailed discussion of the problem in hand.

2. Degree of Convergence of the Series

The study of the degree of convergence of the series (6) involves an examination of the integral

$$\int_0^\pi \phi(x)\, U_n(x)\, dx, \tag{8}$$

which appears in the numerator of the expression (7) for the general coefficient. In evaluating this integral approximately, use will be made of the formula (5) for $U_n(x)$; but this identity will be written at greater length by substituting the whole right-hand side for the function U_n under the sign of integration, and repeating this process a number of times. It will be well to anticipate the discussion of the whole integral (8) by three lemmas relating to the individual terms that will be obtained.

Lemma I. *If $\phi(x)$ has a continuous‡ kth derivative of limited variation in the interval $0 \leqq x \leqq \pi$, and if*

$$\begin{aligned} \phi(0) = \phi'(0) = \cdots = \phi^{(k-1)}(0) \\ = \phi(\pi) = \phi'(\pi) = \cdots = \phi^{(k-1)}(\pi) = 0, \end{aligned} \tag{9}$$

then§

$$\int_0^\pi \phi(x) \cos \rho_n x\, dx = O\left(\frac{1}{n^{k+1}}\right), \tag{10}$$

and

$$\int_0^\pi \phi(x) \sin \rho_n x\, dx = O\left(\frac{1}{n^{k+1}}\right). \tag{11}$$

Consider first the integral containing $\cos \rho_n x$ in the integrand. Since ϕ vanishes at both ends of the interval, integration by parts gives at first

$$\int_0^\pi \phi(x) \cos \rho_n x\, dx = -\frac{1}{\rho_n} \int_0^\pi \phi'(x) \sin \rho_n x\, dx,$$

and after a sufficient number of repetitions,

$$\int_0^\pi \phi(x) \cos \rho_n x\, dx = \frac{1}{\rho_n^k} \int_0^\pi \phi^{(k)}(x) \cos\left(\rho_n x + \frac{k\pi}{2}\right) dx. \tag{12}$$

* See, e. g., Kneser, loc. cit., pp. 109–116, 123–124.

† For a concise exposition of the properties of functions of limited variation that will be used, see, e. g., E. B. Wilson, *Advanced Calculus*, pp. 309, 310.

‡ The assumption of continuity is not necessary, but is made for the sake of convenience.

§ Cf. Picard, loc. cit., where the corresponding proof is given for the case of the Fourier's series.

As the hypothesis is that $\phi^{(k)}(x)$ is of limited variation, let

$$\phi^{(k)}(x) = \phi_1(x) - \phi_2(x), \tag{13}$$

where ϕ_1 and ϕ_2 are positive or zero, continuous, and monotone increasing. By the second law of the mean,

$$\int_0^\pi \phi^{(k)}(x)\cos\left(\rho_n x + \frac{k\pi}{2}\right)dx = \phi_1(\pi)\int_\xi^\pi \cos\left(\rho_n x + \frac{k\pi}{2}\right)dx - \phi_2(\pi)\int_\eta^\pi \cos\left(\rho_n x + \frac{k\pi}{2}\right)dx,$$

where ξ and η are numbers in the interval $(0, \pi)$. The absolute value of the expression just written down does not exceed

$$\frac{2}{\rho_n}[\phi_1(\pi) + \phi_2(\pi)].$$

Since, by (3) and (4), ρ_n is of the same order of magnitude as n, the relation (10) follows at once. The proof of (11) is precisely similar.

If $\phi^{(k)}(0) = 0$, it may be assumed that $\phi_1(0) = \phi_2(0) = 0$, and that the total variation v of $\phi^{(k)}(x)$ is equal to $\phi_1(\pi) + \phi_2(\pi)$. In this case, either of the integrals in the lemma is at most equal to $2v/\rho_n^{k+1}$. It will be convenient to use the letter c a number of times as a general notation for a positive constant, sometimes one and sometimes another, which is independent of n, x, and the function ϕ, though it may depend, for example, on the coefficient λ in the differential equation and on the coefficients h' and H' in the boundary conditions. With this convention, we may say that if $\phi^{(k)}(0) = 0$ each of the integrals remains inferior in absolute value to* cv/n^{k+1}.

Another interesting remark of a special nature is that when $k = 1$ the hypothesis (9), reducing here to

$$\phi(0) = \phi(\pi) = 0,$$

may be abandoned as far as the relation (10) is concerned. For

$$\int_0^\pi \phi(x)\cos\rho_n x\,dx = \frac{1}{\rho_n}[\phi(x)\sin\rho_n x]_0^\pi - \frac{1}{\rho_n}\int_0^\pi \phi'(x)\sin\rho_n x\,dx,$$

and $\sin\rho_n x$ vanishes when $x = 0$, while

$$\sin\rho_n\pi = \pm\sin\epsilon_n\pi = O\left(\frac{1}{n}\right),$$

by (4).

* This is more precise than the original statement of the lemma, because there it was not specified how the constant multiplier implied in the O-notation depends on the function ϕ; here we see that it can be taken proportional to the total variation of $\phi^{(k)}$.

LEMMA II. *If $\phi(x)$ satisfies the conditions of Lemma I, and if $\lambda(x)$ has a continuous $(k-1)$th derivative with limited variation* in $0 \leqq x \leqq \pi$, then*

$$(14) \qquad \int_0^\pi \phi(x) \int_0^x \lambda(t_1) \cos \rho_n t_1 \sin \rho_n (x - t_1)\, dt_1\, dx = O\left(\frac{1}{n^{k+1}}\right),$$

$$(15) \qquad \int_0^\pi \phi(x) \int_0^x \lambda(t_1) \sin \rho_n t_1 \sin \rho_n (x - t_1)\, dt_1\, dx = O\left(\frac{1}{n^{k+1}}\right).$$

By inversion of the order of integration the first of these two integrals may be given the form

$$\int_0^\pi \lambda(t_1) \cos \rho_n t_1 \int_0^\pi \phi(x) \sin \rho_n (x - t_1)\, dx\, dt_1.$$

Let the inner integral here be transformed by integration by parts. The function $\phi(x)$ vanishes with its first $k-1$ derivatives when $x = \pi$, but of course not, in general, when $x = t_1$. At the latter point, the sine of $\rho_n(x - t_1)$ vanishes, while the cosine is equal to 1. Hence

$$\int_{t_1}^\pi \phi(x) \sin \rho_n (x - t_1)\, dx = \frac{1}{\rho_n}\phi(t_1) + \frac{1}{\rho_n}\int_{t_1}^\pi \phi'(x) \cos \rho_n (x - t_1)\, dx$$

$$= \frac{1}{\rho_n}\phi(t_1) - \frac{1}{\rho_n^2}\int_{t_1}^\pi \phi''(x) \sin \rho_n (x - t_1)\, dx,$$

and so on. Finally, if we let $k = 2\gamma$ or $2\gamma - 1$, according as k is even or odd,

$$(16) \qquad \int_{t_1}^\pi \phi(x) \sin \rho_n (x - t_1)\, dx = \frac{1}{\rho_n}\phi(t_1) - \frac{1}{\rho_n^3}\phi''(t_1) + \cdots$$

$$+ \frac{(-1)^{\gamma-1}}{\rho_n^{2\gamma-1}}\phi^{(2\gamma-2)}(t_1) + \frac{1}{\rho_n^k}\int_{t_1}^\pi \phi^{(k)}(x) \sin\left[\rho_n (x - t_1) + \frac{k\pi}{2}\right] dx.$$

By the use of (13) and the second law of the mean, it is seen that the absolute value of the last integral does not exceed

$$\frac{2}{\rho_n}[\phi_1(\pi) + \phi_2(\pi)],$$

whatever the value of t_1 may be.

We have now to multiply the several terms of (16) by $\lambda(t_1) \cos \rho_n t_1$ and integrate with respect to t_1 from 0 to π. The last term, in consequence of what we have just seen, will yield a quantity which is in order of magnitude $O(1/\rho_n^{k+1})$, and therefore $O(1/n^{k+1})$.

* The function $\lambda(x)$ is not subjected to any special restrictions at the ends of the interval.

Consider the first term of the integrated expression,

$$\frac{1}{\rho_n}\int_0^\pi \lambda(t_1)\phi(t_1)\cos\rho_n t_1\,dt_1. \tag{17}$$

The function $\lambda(t_1)\phi(t_1)$ has a $(k-1)$th derivative which is equal to

$$\lambda^{(k-1)}(t_1)\phi(t_1)+(k-1)\lambda^{(k-2)}(t_1)\phi'(t_1)+\cdots+\lambda(t_1)\phi^{(k-1)}(t_1). \tag{18}$$

Now $\phi^{(k-1)}(t_1)$, having a continuous derivative $\phi^{(k)}(t_1)$, is *a fortiori* of limited variation, and the same is true of the earlier derivatives of ϕ and of ϕ itself. Similarly, the derivatives of λ of orders from 0 to $k-2$ are of limited variation as well as the $(k-1)$th. Consequently the property of limited variation is possessed by the whole expression (18). From the corresponding expansions of the earlier derivatives of the product $\lambda\phi$ it is seen that each vanishes at both ends of the interval $(0, \pi)$. It is recognized thus that $\lambda\phi$ satisfies the conditions imposed on ϕ in Lemma I, except that k in the lemma is to be replaced by $k-1$. By that lemma, the integral in (17), without the factor $1/\rho_n$, is $O(1/n^k)$, and when divided by ρ_n becomes $O(1/n^{k+1})$.

Since $\lambda(t_1)\phi''(t_1)$ satisfies the conditions of Lemma I with k replaced by $k-2$, we have

$$\frac{1}{\rho_n^3}\int_0^\pi \lambda(t_1)\phi''(t_1)\cos\rho_n t_1\,dt_1 = O\left(\frac{1}{n^{k+2}}\right).$$

Each of the remaining terms can be disposed of in a similar manner; at each step, from this point on, the number of derivatives of limited variation known to be possessed by the integrand is diminished by 2, and this loss is compensated by the presence of a higher power of $1/\rho_n$ before the integral.

It appears then that (14) is true. The proof of (15) obviously follows the same lines.

Let us look again for a moment at the special case that $\phi^{(k)}(0)=0$. If v still represents the total variation of $\phi^{(k)}(x)$ in $(0, \pi)$,

$$|\phi^{(k)}(x)| \leqq v$$

throughout the interval. Hence the total variation of $\phi^{(k-1)}(x)$ in the interval can not exceed πv. The absolute value of $\phi^{(k-1)}(x)$ can not exceed the same quantity, since $\phi^{(k-1)}(0)=0$. In the same way the total variation of $\phi^{(k-2)}(x)$ and the maximum of its absolute value are less than or equal to $\pi^2 v$, and so on.

It is readily found that the total variation of $\lambda\phi$, for example, does not exceed v multiplied by a quantity independent of ϕ. We can write

$$\lambda\phi=(\lambda_1-\lambda_2)(\phi_1-\phi_2)=(\lambda_1\phi_1+\lambda_2\phi_2)-(\lambda_1\phi_2+\lambda_2\phi_1),$$

where λ_1, λ_2, ϕ_1, and ϕ_2 are monotone increasing and continuous, and the last two vanish at the point 0. It can be assumed also that λ_1 and λ_2 are positive or zero throughout the interval. Then if V is the total variation of $\lambda\phi$,

$$V \leqq \lambda_1(b)\phi_1(b) + \lambda_2(b)\phi_2(b) + \lambda_1(b)\phi_2(b) + \lambda_2(b)\phi_1(b)$$
$$\leqq 2[\lambda_1(b) + \lambda_2(b)]\pi^k v.$$

By similar reasoning, it is found that the $(k-1)$th derivative of $\lambda\phi$, expanded as in (18), and the $(k-2)$th derivative of $\lambda\phi''$, etc., have the same property, namely, that the total variation of each is inferior to v multiplied by a quantity independent of ϕ. Applying this fact in the preceding proof of Lemma II, we conclude that when $\phi^{(k)}(0) = 0$ the statement of the lemma can be made more precise by saying that the absolute value of each of the two integrals concerned remains inferior to cv/n^{k+1}.

LEMMA III. *If ϕ and λ satisfy the conditions of Lemma II, then**

$$\int_0^\pi \phi(x) \int_0^x \lambda(t_1) \sin \rho_n(x - t_1) \int_0^{t_1} \lambda(t_2) \sin \rho_n(t_1 - t_2) \int_0^{t_2} \cdots$$
$$\int_0^{t_{s-1}} \lambda(t_s) \cos \rho_n t_s \sin \rho_n(t_{s-1} - t_s)\, dt_s\, dt_{s-1} \cdots dt_1\, dx = O\left(\frac{1}{n^{k+1}}\right), \tag{19}$$

and the same is true if $\cos \rho_n t_s$ *in the innermost integral is replaced by* $\sin \rho_n t_s$.

The proof is obtained by induction. The statement is already known to be true when $s = 1$, as it reduces then to Lemma II, the variable x taking the place of a variable t_0. We shall assume that the conclusion holds if s is replaced by $s - 1$, and show then that it holds as stated. The steps in the passage from $s - 1$ to s correspond exactly to those carried out in the proof of Lemma II. We begin by inverting the order of integration, confining our attention to the integral with $\cos \rho_n t_s$. The given expression is equal to

$$\int_0^\pi \lambda(t_s) \cos \rho_n t_s \int_{t_s}^\pi \lambda(t_{s-1}) \sin \rho_n(t_{s-1} - t_s) \int_{t_{s-1}}^\pi \cdots$$
$$\int_{t_2}^\pi \lambda(t_1) \sin \rho_n(t_1 - t_2) \int_{t_1}^\pi \phi(x) \sin \rho_n(x - t_1)\, dx\, dt_1 \cdots dt_{s-1}\, dt_s.$$

The innermost integral here is precisely that which appears in (16). By the aid of that formula, we express the $(s+1)$-fold integral as the sum of $\gamma + 1$ terms, of which the last is immediately seen to be $O(1/n^{k+1})$, while the others

* In this multiple integral, the notation for which is suggestive rather than complete, $s - 3$ more factors of the form $\lambda(t_i) \sin \rho_n(t_{i-1} - t_i)$ are to be understood. It will be seen a few lines below how an integral of this form is obtained as a result of successive substitutions in (5).

are shown to be $O(1/n^{k+1})$, or even $O(1/n^{k+2})$, by means of the assumed lemma for s-fold integrals, with ϕ replaced successively by $\lambda\phi$, $\lambda\phi''$, $\cdots$, $\lambda\phi^{(2\gamma-2)}$. Thus the relation (19) is established, and clearly remains true if $\sin \rho_n t_s$ is written instead of $\cos \rho_n t_s$.

In the special case that $\phi^{(k)}(0) = 0$, it is readily shown by the use of the corresponding refinement of Lemma II that each of the two integrals in the present lemma remains inferior in absolute value to cv/n^{k+1}.

We are now in a position to deal with the integral (8). If, on the right-hand side of (5), the function $U_n(t)$ under the integral sign is expressed at length by means of (5) itself, the following more extended formula results:

$$
\begin{aligned}
U_n(x) = \cos \rho_n x + \frac{h' \sin \rho_n x}{\rho_n} + \frac{1}{\rho_n}\int_0^x \lambda(t_1)\cos \rho_n t_1 \sin \rho_n (x - t_1)\, dt_1 \\
+ \frac{h'}{\rho_n^2}\int_0^x \lambda(t_1)\sin \rho_n t_1 \sin \rho_n (x - t_1)\, dt_1 \qquad (20) \\
+ \frac{1}{\rho_n^2}\int_0^x \lambda(t_1)\sin \rho_n (x - t_1)\int_0^{t_1} \lambda(t_2)\, U_n(t_2)\sin \rho_n (t_1 - t_2)\, dt_2\, dt_1.
\end{aligned}
$$

There is still an integral on the right-hand side containing U_n under the integral sign, and the step just taken can be repeated. This process is to be continued until an expression is obtained in which the integral involving U_n is preceded by a factor $1/\rho_n^{k+1}$, where k is an appropriate integer. Let this expression be multiplied by $\phi(x)$ and integrated from 0 to π with regard to x. The last of the resulting terms will be $O(1/n^{k+1})$, inasmuch as the integral, without the factor $1/\rho_n^{k+1}$, remains finite for all values of n. The integrals which appear in the earlier terms are explicit expressions of the types considered in Lemmas I, II, and III. If ϕ and λ are restricted as in those lemmas, each integral, even without the power of $1/\rho_n$ which stands before it, will be $O(1/n^{k+1})$. If λ is supposed provided with only $k-2$ continuous derivatives of limited variation, instead of $k-1$, so that k is to be replaced by $k-1$ in applying Lemmas II and III, each term will still be $O(1/n^{k+1})$ after the factors $1/\rho_n$ have been multiplied in. The conclusion may be stated thus:

LEMMA IV. *If ϕ satisfies the conditions of Lemma I, and if λ has a continuous $(k-2)$th derivative of limited variation in $0 \leqq x \leqq \pi$, then*

$$
\int_0^\pi \phi(x)\, U_n(x)\, dx = O\left(\frac{1}{n^{k+1}}\right). \qquad (21)
$$

If, in addition, λ has a continuous $(k-1)$th derivative of limited variation in the interval,

$$
\int_0^\pi \phi(x)\, U_n(x)\, dx = \int_0^\pi \phi(x)\cos \rho_n x\, dx + O\left(\frac{1}{n^{k+2}}\right). \qquad (22)
$$

If $\phi^{(k)}(0) = 0$, the constant multiplier implied in the O-notation can be taken as v, the total variation of $\phi^{(k)}(x)$, multiplied by a quantity independent* of ϕ.

Let the case $k = 1$ be brought up again for special consideration at this point. The hypothesis for the first part of Lemma IV takes the form that ϕ has a continuous first derivative of limited variation and that

$$\phi(0) = \phi(\pi) = 0,$$

while nothing more than continuity is required of λ. The point to be made is that in this case the special restriction on ϕ at the ends of the interval is unnecessary. Let $U_n(x)$ be expanded to five terms as in (20), multiplied by $\phi(x)$, and integrated. It has already been pointed out, in connection with the proof of Lemma I, that

$$\int_0^\pi \phi(x) \cos \rho_n x \, dx = O\left(\frac{1}{n^2}\right),$$

whether ϕ vanishes at the ends of the interval or not. As for the other terms, the application of Lemma II with $k = 0$ gives all that is required.†

The discussion so far has related to the numerator of the expression (7) for α_n. For the denominator we need only the simple relation given by Liouville:‡

$$\int_0^\pi [U_n(x)]^2 \, dx = \frac{\pi}{2} + O\left(\frac{1}{n}\right). \tag{23}$$

The proof is simply this: In consequence of (5),

$$\int_0^\pi [U_n(x)]^2 \, dx = \int_0^\pi \cos^2 \rho_n x \, dx + O\left(\frac{1}{n}\right),$$

and

$$\int_0^\pi \cos^2 \rho_n x \, dx = \frac{\pi}{2} + \frac{\sin 2\rho_n \pi}{4\rho_n} = \frac{\pi}{2} + O\left(\frac{1}{n}\right).$$

We have the materials now for one theorem on the degree of convergence

* It is to be noticed that the formulas on which all our demonstrations depend hold only for values of n *from a certain point on*, but it is readily seen that the conclusions are correct for all positive values of n. This remark is not trivial in the case of such a proposition as the one to which this note is appended. It is asserted, for example, that

$$\int_0^\pi \phi(x) \, U_1(x) \, dx$$

does not exceed a constant multiple of v; this is true with the hypotheses that are stated, since the maximum of $|\phi|$ does not exceed a constant multiple of v, but would not generally be true otherwise.

† If this lemma and (20) had not already been written out at length, it would have been still simpler to operate with the three-term expression (5) directly.

‡ Loc. cit., p. 430.

of the series (6). If ϕ and λ satisfy the hypotheses for the first part of Lemma IV, it follows from (21) and (23) that

$$\alpha_n = O\left(\frac{1}{n^{k+1}}\right),$$

and as the maximum of $|U_n(x)|$ remains finite when n becomes infinite,

$$|\alpha_n U_n(x)| \leqq \frac{\beta}{n^{k+1}},$$

where β is independent of n and x. It follows that the series $\sum \alpha_n U_n(x)$ converges uniformly,* in which case, as has already been stated, it must converge to the value $\phi(x)$. If we set

$$\sigma_n(x) = \sum_{\nu=0}^{n} \alpha_\nu U_\nu(x),$$

then

$$|\phi(x) - \sigma_n(x)| = \left| \sum_{\nu=n+1}^{\infty} \alpha_\nu U_\nu(x) \right| \leqq \sum_{\nu=n+1}^{\infty} \frac{\beta}{\nu^{k+1}} < \beta \int_n^\infty \frac{dt}{t^{k+1}} = \frac{\beta}{kn^k}.$$

That is:

THEOREM I. *If $\phi(x)$ has a continuous kth derivative of limited variation in the interval $0 \leqq x \leqq \pi$, while ϕ itself and its first $k-1$ derivatives vanish for $x = 0$ and for $x = \pi$, and if, furthermore, the function $\lambda(x)$ which appears as a coefficient in the differential equation has a continuous $(k-2)$th derivative of limited variation in $0 \leqq x \leqq \pi$, then*

$$\phi(x) = \sigma_n(x) + O\left(\frac{1}{n^k}\right)$$

uniformly throughout the interval.

Whenever the O-notation is used in the present paper in a relation involving a function of x, it will be understood, without being repeated on every occasion, that the relation holds uniformly, that is, that the implicit constant multiplier is independent of x as well as of n. In the relation just written down it depends on k, λ, h', H', and ϕ. In the case that $\phi^{(k)}(0) = 0$ its dependence on ϕ is completely characterized by saying that it is proportional to the total variation of $\phi^{(k)}$.

When $k = 1$, the theorem is true without the restriction that

$$\phi(0) = \phi(\pi) = 0,$$

in consequence of the facts previously established for this special case. That the restrictions at the ends of the interval are not superfluous in general is shown by the simplest examples. If

$$\lambda(x) = 0, \qquad h' = H' = 0, \qquad \phi(x) = \sin x,$$

* Of course this is assured at the outset by well-known general theorems, but it appears incidentally here.

for instance, so that the series is the cosine-series for $\sin x$, the hypotheses of Theorem I, except those relating to the end-points, are satisfied for any value of k, but the remainder in the series

$$\sin x = \frac{4}{\pi}\left[1 - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \cdots - \frac{\cos 2nx}{4n^2-1} - \cdots\right]$$

at the point $x = \frac{1}{2}\pi$ does not approach zero faster than $1/n^2$.

A further theorem can be obtained by operating with (22). Let the equation (12) be recalled, and compared with the following, which is obtained in the same way:

$$\int_0^\pi \phi(x)\cos nx\,dx = \frac{1}{n^k}\int_0^\pi \phi^{(k)}(x)\cos\left(nx + \frac{k\pi}{2}\right)dx.$$

By (3) and (4) we may write

$$\frac{1}{\rho_n^k} = \frac{1}{n^k}\left(\frac{n}{\rho_n}\right)^k = \frac{1}{n^k}\left(1 - \frac{\epsilon_n}{\rho_n}\right)^k = \frac{1}{n^k} + O\left(\frac{1}{n^{k+2}}\right).$$

On the other hand,

$$\cos\left(\rho_n x + \frac{k\pi}{2}\right) = \cos\left(nx + \frac{k\pi}{2}\right) - (1 - \cos\epsilon_n x)\cos\left(nx + \frac{k\pi}{2}\right) - \sin\epsilon_n x \sin\left(nx + \frac{k\pi}{2}\right)$$

$$= \cos\left(nx + \frac{k\pi}{2}\right) - \sin\epsilon_n x \sin\left(nx + \frac{k\pi}{2}\right) + O\left(\frac{1}{n^2}\right).$$

Now $\sin\epsilon_n x$ is a positive monotone increasing function in $(0, \pi)$, as soon as n is sufficiently large, and so, if we write

$$\phi^{(k)}(x) = \phi_1(x) - \phi_2(x),$$

as in (13), we shall have

$$\int_0^\pi \phi^{(k)}(x)\sin\epsilon_n x \sin\left(nx + \frac{k\pi}{2}\right)dx$$

$$= \phi_1(\pi)\sin\epsilon_n\pi\int_\xi^\pi \sin\left(nx + \frac{k\pi}{2}\right)dx - \phi_2(\pi)\sin\epsilon_n\pi\int_\eta^\pi \sin\left(nx + \frac{k\pi}{2}\right)dx$$

$$= O\left(\frac{1}{n^2}\right),$$

since $\sin\epsilon_n\pi$ and both the integrals are separately $O(1/n)$. By combination of these relations,

$$\frac{1}{\rho_n^k}\int_0^\pi \phi^{(k)}(x)\cos\left(\rho_n x+\frac{k\pi}{2}\right)dx$$

$$=\left[\frac{1}{n^k}+O\left(\frac{1}{n^{k+2}}\right)\right]\left[\int_0^\pi \phi^{(k)}(x)\cos\left(nx+\frac{k\pi}{2}\right)dx+O\left(\frac{1}{n^2}\right)\right]$$

$$=\frac{1}{n^k}\int_0^\pi \phi^{(k)}(x)\cos\left(nx+\frac{k\pi}{2}\right)dx+O\left(\frac{1}{n^{k+2}}\right).$$

That is,

$$\int_0^\pi \phi(x)\cos\rho_n x\,dx=\int_0^\pi \phi(x)\cos nx\,dx+O\left(\frac{1}{n^{k+2}}\right);$$

the relation (22) remains true if $\cos\rho_n x$ is replaced by $\cos nx$. The integral on the right is itself $O(1/n^{k+1})$. From (23),

$$\frac{1}{\int_0^\pi [U_n(x)]^2\,dx}=\frac{2}{\pi}+O\left(\frac{1}{n}\right).$$

Hence, by multiplication,

$$\alpha_n=\frac{2}{\pi}\int_0^\pi \phi(x)\cos nx\,dx+O\left(\frac{1}{n^{k+2}}\right).$$

To go one step further,

$$U_n(x)=\cos\rho_n x+O\left(\frac{1}{n}\right)=\cos nx+O\left(\frac{1}{n}\right),$$

and so

$$\alpha_n U_n(x)=\frac{2}{\pi}\cos nx\int_0^\pi \phi(t)\cos nt\,dt+O\left(\frac{1}{n^{k+2}}\right). \tag{24}$$

Up to this point the original hypotheses about the function ϕ have been retained. It is for a somewhat different class of functions that the last relations are to be used. Suppose that $\phi(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition,

$$|\phi^{(k-1)}(x_2)-\phi^{(k-1)}(x_1)|\leqq\mu|x_2-x_1|, \tag{25}$$

where x_1 and x_2 are any two values in the closed interval $(0,\pi)$ and μ is a constant. This derivative is *a fortiori* continuous and of limited variation, and if we suppose that ϕ and its first $k-2$ derivatives vanish at 0 and π, and that λ has a continuous derivative of order $k-2$ with limited variation, the hypotheses for (24) will be fulfilled with k replaced by $k-1$.

The function $\phi(x)$ has been defined up to the present only in the interval from 0 to π. Let this definition be extended by setting $\phi(x)=\phi(-x)$ for $-\pi\leqq x\leqq 0$, and then making $\phi(x+2\pi)=\phi(x)$ for all real values of x. The periodic function so defined will have a $(k-1)$th derivative

satisfying a Lipschitz condition everywhere* if k is odd, and this will be true for even values of k as well if the hypotheses previously made are supplemented by requiring that $\phi^{(k-1)}(0) = \phi^{(k-1)}(\pi) = 0$.

The Fourier's series for $\phi(x)$ will involve only cosine-terms, since ϕ is even, and will have for its general term precisely the expression on the right-hand side of (24), with the O-term omitted. Let this general term of the Fourier's series be denoted by $a_n \cos nx$, the sum of the first $n+1$ terms by $s_n(x)$. It is known that†

$$\phi(x) - s_n(x) = \sum_{\nu=n+1}^{\infty} a_\nu \cos \nu x = O\left(\frac{\log n}{n^k}\right). \tag{26}$$

Remembering that k in (24) is to be replaced by $k-1$, we have

$$|\alpha_n U_n(x) - a_n \cos nx| \leqq \frac{\beta}{n^{k+1}}, \tag{27}$$

where β is independent of n and x. It is understood that‡ $k \geqq 1$; it appears from the relations just written down that the Sturm-Liouville series $\sum \alpha_n U_n(x)$ converges uniformly, and that

$$\begin{aligned}
|\phi(x) - \sigma_n(x)| &\leqq |\phi(x) - s_n(x)| + |[\phi(x) - \sigma_n(x)] - [\phi(x) - s_n(x)]| \\
&= |\phi(x) - s_n(x)| + \left| \sum_{\nu=n+1}^{\infty} [\alpha_\nu U_\nu(x) - a_\nu \cos \nu x] \right| \\
&\leqq |\phi(x) - s_n(x)| + \sum_{\nu=n+1}^{\infty} \frac{\beta}{\nu^{k+1}} \\
&= O\left(\frac{\log n}{n^k}\right) + O\left(\frac{1}{n^k}\right) = O\left(\frac{\log n}{n^k}\right).
\end{aligned}$$

This may be formulated as follows:

THEOREM II. *If $\phi(x)$ has a $(k-1)$th derivative satisfying a Lipschitz condition throughout the interval $0 \leqq x \leqq \pi$, while ϕ itself and its first $k-2$ derivatives, and, in case k is even, the $(k-1)$th derivative also, vanish for*

* It is seen at once that ϕ itself is continuous, and readily shown that the successive derivatives through the $(k-2)$th exist and are continuous, even at the points $0, \pi, 2\pi \cdots$. The $(k-2)$th derivative will be an even or an odd function according as k is even or odd. In the latter case its right-hand and left-hand derivatives at the point 0 will be equal; in the former they will be the negatives of each other, and so equal only if they vanish. Similar reasoning applies to the point π. The others need not be considered separately, because of the periodicity of ϕ. The proof that the Lipschitz condition is satisfied offers no difficulty.

† For the case $k = 1$, see Lebesgue, loc. cit., p. 201; for the general case, D. Jackson, loc. cit., also D. Jackson, these Transactions, vol. 14 (1913), pp. 343–364, Theorem X. The latter paper will be referred to as B.

‡ The meaning of the hypothesis in the case $k = 1$ would be, of course, that ϕ itself satisfies a Lipschitz condition.

$x = 0$ and for $x = \pi$, and if $\lambda(x)$ has a continuous $(k-2)$th derivative of limited variation for $0 \leqq x \leqq \pi$, then

$$\phi(x) = \sigma_n(x) + O\left(\frac{\log n}{n^k}\right)$$

uniformly throughout the interval.

If k is even, so that it is assumed that $\phi^{(k-1)}(0) = 0$, or if k is odd and this assumption is added to those already made, it will be seen, on following through the work with this in view, that the remainder does not exceed $c\mu(\log n)/n^k$, where c is independent of ϕ, and μ is the coefficient in the Lipschitz condition which $\phi^{(k-1)}(x)$ satisfies,* and it is assumed that $n \geqq 2$. It is essential for this purpose to note from the papers referred to for the Fourier's series that the constant factor implicit in the O-symbol in (26) may be taken as μ multiplied by a quantity independent of ϕ.

The suggested refinement of Theorem II may be dismissed with these few lines, as far as general values of k are concerned. In the simplest case, $k = 1$, the theorem is needed for an application in its more precise form, and it will be well to state this simple result separately and to give the details of the proof from the beginning.†

THEOREM II*a*. *If $\phi(x)$ satisfies the Lipschitz condition*

$$|\phi(x_2) - \phi(x_1)| \leqq \mu|x_2 - x_1| \tag{28}$$

throughout the interval $0 \leqq x \leqq \pi$, and if $\phi(0) = 0$, then, in the whole interval,

$$|\phi(x) - \sigma_n(x)| \leqq \frac{c\mu \log n}{n}, \qquad n \geqq 2,$$

where c is independent of x, n, and ϕ.

The restriction on $\lambda(x)$ is merely that of continuity.

In accordance with the hypothesis, $|\phi(x)| \leqq \pi\mu$, and $\phi(x)$ can be written in the form

$$\phi(x) = \phi_1(x) - \phi_2(x), \tag{29}$$

where ϕ_1 and ϕ_2 are continuous and monotone increasing, and

$$\phi_1(0) = \phi_2(0) = 0, \qquad \phi_1(\pi) \leqq \pi\mu, \qquad \phi_2(\pi) \leqq \pi\mu.$$

The expression (5) for $U_n(x)$ will be sufficient for our purpose. Let the second term be multiplied by $\phi(x)$ and integrated; we find

* It is obvious that the total variation of $\phi^{(k-1)}(x)$ in $(0, \pi)$ is at most $\pi\mu$.

† This will involve some repetition of what has gone before, which is perhaps compensated by the gain in clearness.

$$\left|\int_0^\pi \phi(x)\sin\rho_n x\,dx\right| = \left|\phi_1(\pi)\int_\xi^\pi \sin\rho_n x\,dx - \phi_2(\pi)\int_\eta^\pi \sin\rho_n x\,dx\right|$$
$$\leqq \frac{4\pi\mu}{\rho_n} \leqq \frac{c\mu}{n},$$

and hence

$$\left|\frac{h'}{\rho_n}\int_0^\pi \phi(x)\sin\rho_n x\,dx\right| \leqq \frac{c\mu}{n^2};$$

as has already been explained, the letter c will be used to represent a number independent of n and ϕ, and, in a relation where x occurs, independent also of x, but will stand for different numbers of this sort in different lines, and, on occasion, even in the same line. Consider the third term:

$$\int_0^\pi \phi(x)\int_0^\pi \lambda(t)\,U_n(t)\sin\rho_n(x-t)\,dt\,dx$$
$$= \int_0^\pi \lambda(t)\,U_n(t)\cos\rho_n t\int_t^\pi \phi(x)\sin\rho_n x\,dx\,dt$$
$$- \int_0^\pi \lambda(t)\,U_n(t)\sin\rho_n t\int_t^\pi \phi(x)\cos\rho_n x\,dx\,dt,$$

and as each of the integrals extended from t to π is seen by the use of (29) and the second law of the mean to be in absolute value not greater than $c\mu/n$, it follows that the whole expression satisfies an inequality of the same form. Consequently

$$\left|\frac{1}{\rho_n}\int_0^\pi \phi(x)\int_0^\pi \lambda(t)\,U_n(t)\sin\rho_n(x-t)\,dt\,dx\right| \leqq \frac{c\mu}{n^2}.$$

Return now to the first term of (5); here it is to be pointed out that

$$\left|\int_0^\pi \phi(x)\cos\rho_n x\,dx - \int_0^\pi \phi(x)\cos nx\,dx\right|$$
$$\leqq \left|\int_0^\pi (\cos\epsilon_n x - 1)\,\phi(x)\cos nx\,dx\right| + \left|\int_0^\pi \phi(x)\sin\epsilon_n x\sin nx\,dx\right|$$
$$\leqq \frac{c\mu}{n^2} + \left|\phi_1(\pi)\sin\epsilon_n\pi\int_\xi^\pi \sin nx\,dx - \phi_2(\pi)\sin\epsilon_n\pi\int_\eta^\pi \sin nx\,dx\right|$$
$$\leqq \frac{c\mu}{n^2},$$

the last inequality but one being obtained as soon as n is so large that* $\epsilon_n \leqq \frac{1}{2}$ and $\sin\epsilon_n x$ is monotone increasing in $(0,\pi)$. To sum up the inequalities obtained thus far,

$$(30)\qquad \int_0^\pi \phi(x)\,U_n(x)\,dx = \int_0^\pi \phi(x)\cos nx\,dx + r_1(n,x),$$

* The point where this begins to be true is of course independent of ϕ.

where the remainder r_1 is such that

$$|r_1(n, x)| \leqq \frac{c\mu}{n^2}.$$

In consequence of (23), which was proved in four lines, we may write

$$\frac{1}{\int_0^\pi [U_n(x)]^2\, dx} = \frac{2}{\pi} + r_2(n), \tag{31}$$

where

$$|r_2(n)| \leqq \frac{c}{n}.$$

Furthermore,

$$U_n(x) = \cos \rho_n x + r_3(n, x) = \cos nx + r_4(n, x), \tag{32}$$

where

$$|r_3(n, x)| \leqq \frac{c}{n}, \qquad |r_4(n, x)| \leqq \frac{c}{n}.$$

As the second law of the mean shows that

$$\left| \int_0^\pi \phi(x) \cos nx\, dx \right| \leqq \frac{c\mu}{n},$$

multiplication of (30), (31), and (32) gives

$$\alpha_n U_n(x) = \frac{2}{\pi} \cos nx \int_0^\pi \phi(t) \cos nt\, dt + r_5(n, x), \tag{33}$$

where

$$|r_5(n, x)| \leqq \frac{c\mu}{n^2}. \tag{34}$$

Let $\phi(x)$ be defined outside of $(0, \pi)$ so as to be an even function of period 2π for all real values of x. This function will satisfy (28) everywhere. If $a_n \cos nx$ is the general term of its Fourier's series, which contains no sine-terms, and

$$s_n(x) = \sum_{\nu=0}^{n} a_n \cos nx,$$

then*

$$|\phi(x) - s_n(x)| \leqq \frac{c\mu \log n}{n}, \qquad n \geqq 2.$$

The relation (33), with (34), states that

$$|\alpha_n U_n(x) - a_n \cos nx| \leqq \frac{c\mu}{n^2}.$$

* Lebesgue, loc. cit., p. 201; D. Jackson, in the paper A, Theorem V.

This has been established only for values of n from a certain point on, this point being unspecified, but independent of ϕ. On taking account of the latter circumstance, it is readily seen that the relation holds also for the positive values of n previously neglected. For the inequalities

$$|\alpha_n U_n(x)| \leqq c\mu, \qquad |a_n \cos nx| \leqq c\mu,$$

are seen at once to hold for all these values of n, since $|\phi(x)| \leqq \pi\mu$, and $U_n(x)$, whether represented by (5) or not, is continuous and not identically zero; it follows that

$$|\alpha_n U_n(x) - a_n \cos nx| \leqq c\mu,$$

and it is immaterial whether we write $c\mu$ or $c\mu/n^2$, since only a finite number of values of n are concerned. Consequently, for $n \geqq 2$,

$$\begin{aligned}|\phi(x) - \sigma_n(x)| &\leqq |\phi(x) - s_n(x)| + |[\phi(x) - \sigma_n(x)] - [\phi(x) - s_n(x)]| \\ &= |\phi(x) - s_n(x)| + \left| \sum_{\nu=n+1}^{\infty} [\alpha_\nu U_\nu(x) - a_\nu \cos \nu x] \right| \\ &\leqq |\phi(x) - s_n(x)| + \sum_{\nu=n+1}^{\infty} \frac{c\mu}{n^2} \\ &\leqq \frac{c\mu \log n}{n} + \frac{c\mu}{n} \leqq \frac{c\mu \log n}{n}.\end{aligned}$$

Now let $\phi(x)$ be any function whatever that is continuous in $0 \leqq x \leqq \pi$ and vanishes for $x = 0$. Represent by δ any positive quantity, and let $\omega(\delta)$ denote the maximum of $|\phi(x_2) - \phi(x_1)|$ for values of x_1 and x_2 in the interval subject to the restriction that $|x_2 - x_1| \leqq \delta$. It is obvious at once that $\omega(\delta)$ is a monotone increasing function of δ, and that

$$\lim_{\delta=0} \omega(\delta) = 0.$$

It may be assumed further that $\omega(\delta)/\delta$ does not approach zero as δ approaches zero, otherwise ϕ would have a vanishing derivative throughout the interval and so would vanish identically itself. The relation just obtained is to be used to establish the following proposition:

THEOREM III. *If $\phi(x)$ is continuous in $0 \leqq x \leqq \pi$ and*

$$|\phi(x_2) - \phi(x_1)| \leqq \omega(\delta) \tag{35}$$

whenever

$$|x_2 - x_1| \leqq \delta,$$

and if $\phi(0) = 0$, then

$$|\phi(x) - \sigma_n(x)| \leqq c\omega\left(\frac{\pi}{n}\right) \log n, \qquad n \geqq 2.$$

The proof, after Theorem II*a* has been established, is that given by Lebesgue* for the corresponding property of Fourier's series. We begin by deriving the following lemma:

LEMMA V. *If*

$$|\phi(x)| \leqq \epsilon$$

throughout the interval $0 \leqq x \leqq \pi$, *then*

$$|\phi(x) - \sigma_n(x)| \leqq c\epsilon \log n, \qquad n \geqq 2.$$

It is obviously sufficient to show that $|\sigma_n(x)|$ itself can not exceed $c\epsilon \log n$. Now it follows at once from (5), with (3) and (4), that

$$\left| \int_0^\pi \phi(x)\, U_n(x)\, dx - \int_0^\pi \phi(x) \cos \rho_n x\, dx \right| \leqq \frac{c\epsilon}{n},$$

or, since $|\cos \rho_n x - \cos nx| \leqq c/n$, that

$$\left| \int_0^\pi \phi(x)\, U_n(x)\, dx - \int_0^\pi \phi(x) \cos nx\, dx \right| \leqq \frac{c\epsilon}{n}.$$

From this and (31) and (32),

$$|\alpha_n U_n(x) - a_n \cos nx| \leqq \frac{c\epsilon}{n}.$$

Therefore

$$|\sigma_n(x) - s_n(x)| \leqq \sum_{\nu=1}^{n} \frac{c\epsilon}{\nu} \leqq c\epsilon \log n,$$

and as it is well known† that

$$|s_n(x)| \leqq c\epsilon \log n,$$

the lemma follows at once.

Now let a function $\psi(x)$ be defined as equal to $\phi(x)$ at the points $x = i\pi/n$, $i = 0, 1, \cdots, n$, and linear between these points. Because of (35),

$$|\phi(x) - \psi(x)| \leqq 2\omega\left(\frac{\pi}{n}\right) \tag{36}$$

throughout the interval. Furthermore, $\psi(x)$ satisfies (28) with

$$\mu = \frac{\omega(\pi/n)}{\pi/n}. \tag{37}$$

Let us denote by $\sigma_n(x)$ the sum of the first $n+1$ terms of the Sturm-Liouville series for $\phi(x)$, and by $\sigma_{1n}(x)$ and $\sigma_{2n}(x)$ the corresponding sums formed for $\psi(x)$ and $\phi(x) - \psi(x)$ respectively. Of course

$$\sigma_n(x) = \sigma_{1n}(x) + \sigma_{2n}(x).$$

* Loc. cit., pp. 201–202.

† See, e. g., Lebesgue, loc. cit., pp. 196–197.

By (37) and Theorem II*a*,

$$|\psi(x) - \sigma_{1n}(x)| \leqq \frac{c \log n}{n} \cdot \frac{\omega(\pi/n)}{\pi/n} = c\omega\left(\frac{\pi}{n}\right)\log n.$$

By (36) and Lemma V,

$$|\sigma_{2n}(x)| \leqq c\omega\left(\frac{\pi}{n}\right)\log n.$$

As it follows from (36) *a fortiori* that

$$|\phi(x) - \psi(x)| \leqq c\omega\left(\frac{\pi}{n}\right)\log n,$$

the identity

$$\phi(x) - \sigma_n(x) = \phi(x) - \psi(x) + \psi(x) - \sigma_{1n}(x) - \sigma_{2n}(x)$$

gives the demonstration of Theorem III at once.

If $\phi(x)$ is continuous, but $\phi(0) \neq 0$, let

$$\phi(x) = \phi(0) + \chi(x).$$

Theorem I is applicable to the Sturm-Liouville series for the constant $\phi(0)$, with $k = 1$; for of course a constant has a first derivative of limited variation, and it was pointed out in connection with that theorem that for $k = 1$ the function developed need not vanish at the ends of the interval. Hence the remainder after n terms of the series is $O(1/n)$. On the other hand, Theorem III may be applied to the function $\chi(x)$. If $\omega(\delta)$, formed for this function, is such that the ratio of δ to $\omega(\delta)$ remains finite as δ approaches zero,* then

$$\frac{1}{n} = O\left[\omega\left(\frac{\pi}{n}\right)\right] = O\left[\omega\left(\frac{\pi}{n}\right)\log n\right],$$

and $\phi(x)$ still has the property that

$$\phi(x) = \sigma_n(x) + O\left[\omega\left(\frac{\pi}{n}\right)\log n\right],$$

but it is no longer true that the constant multiplier in the O-symbol is independent of ϕ.

3. Summation of the Series

The theorem expressed in (26), concerning the degree of convergence of the Fourier's series for a function $\phi(x)$ of period 2π having a $(k-1)$th derivative that satisfies a Lipschitz condition, was proved by an indirect method. It was obtained as a consequence of the two following propositions:

(*a*) If $\phi(x)$ is a function of the character described, it is possible to define for each positive integral value of n a finite trigonometric sum of the nth order

* This assumption goes somewhat beyond the earlier one, that $\omega(\delta)/\delta$ shall not approach zero, that is, that the ratio of δ to $\omega(\delta)$ shall not become infinite.

at most which represents $\phi(x)$ approximately with a maximum error not exceeding a constant multiple of* $1/n^k$.

(*b*) If $\phi(x)$ is any function† of period 2π which can be approximately represented by a finite trigonometric sum of the nth or lower order with an error nowhere exceeding ϵ, and if $s_n(x)$ is the corresponding partial sum of the Fourier's series for $\phi(x)$, then, for $n \geqq 2$,

$$|\phi(x) - s_n(x)| \leqq K\epsilon \log n,$$

where K is an absolute constant.‡

It may not be unprofitable to observe that each of these propositions, which are interesting for their own sake, has its analogue in the theory of the Sturm-Liouville series. For (*b*) this analogue reads as follows:

If $\phi(x)$ can be approximately represented throughout the closed interval $(0, \pi)$ by a linear combination $\Sigma_n(x)$ of the characteristic functions $U_0(x)$, $U_1(x)$, $\cdots$, $U_n(x)$, with constant coefficients, with an error nowhere exceeding ϵ, and if $\sigma_n(x)$ denotes the corresponding partial sum of the Sturm-Liouville series for $\phi(x)$, then, for $n \geqq 2$,

$$|\phi(x) - \sigma_n(x)| \leqq c\epsilon \log n$$

throughout the interval.

Here, as everywhere in this paper, c denotes a constant independent of ϕ, but depending conceivably on $\lambda(x)$ and on the coefficients in the boundary conditions.

The truth of the assertion is an immediate consequence of Lemma V. The partial sum of the Sturm-Liouville series for $\phi(x)$ is obtained by adding those for $\Sigma_n(x)$ and for $\phi(x) - \Sigma_n(x)$; the partial sum for $\Sigma_n(x)$ is $\Sigma_n(x)$ itself, and an error can arise only in the partial sum for $\phi(x) - \Sigma_n(x)$, to which Lemma V applies.

The generalization of (*a*) is somewhat less trivial. Let us recall the method by which that theorem itself was proved in the paper A.§

If k is the integer which appears in the statement of the theorem, and n is any positive integer, let κ be the smallest integer for which $2\kappa - k > 1$, and m the largest integer for which $\kappa(m-1)$ does not exceed n, and let

$$S_n(x) = h_m \int_{-\pi/2}^{\pi/2} [(-1)^{k+1}\phi(x+2ku) + (-1)^k k\phi(x+2(k-1)u) + \cdots + k\phi(x+2u)]\left[\frac{\sin mu}{m \sin u}\right]^{2\kappa} du, \tag{38}$$

* D. Jackson, Dissertation, Göttingen 1911, Theorem VII; A, Theorem III; B, Theorem III.

† Integrable in the sense of Lebesgue.

‡ Lebesgue, loc. cit., p. 201; also, earlier, Lebesgue, Annales de la Faculté des Sciences de l'Université de Toulouse pour les sciences mathématiques et les sciences physiques, series 3, vol. 1 (1909), pp. 25–117; pp. 116–117.

§ The neater proof given in B seems less convenient for the present purpose.

where the numerical coefficients in the integrand are the binomial coefficients corresponding to the exponent k, the last being omitted, and h_m is a constant defined by the equation

$$\frac{1}{h_m} = \int_{-\pi/2}^{\pi/2} \left[\frac{\sin mu}{m \sin u}\right]^{2\kappa} du.$$

It is shown in A* that $S_n(x)$ is a trigonometric sum in x of order not higher than $\kappa(m-1)$, and so not higher than n, and that $S_n(x)$ represents $\phi(x)$ with the required degree of approximation.

This information as to the form of $S_n(x)$ will not be enough for us here. We shall show that†

$$S_n(x) = \int_{-\pi}^{\pi} \phi(t)\, T_n(t-x)\, dt, \tag{39}$$

where

$$T_n(t-x) = \sum_{\nu=0}^{n} A_{n\nu} \cos \nu(t-x), \tag{40}$$

and the coefficients $A_{n\nu}$ are constants independent of ϕ. It will be sufficient to show that each of the k terms of which $S_n(x)$ is composed in (38) has this form. Consider any one of these terms, say that containing $\phi(x+2ru)$. The binomial coefficient and the factor $\pm h_m$ may be left out of account. The change of variable $t = x + 2ru$ gives

$$\int_{-\pi/2}^{\pi/2} \phi(x+2ru) \left[\frac{\sin mu}{m \sin u}\right]^{2\kappa} du = \frac{1}{2r} \int_{x-r\pi}^{x+r\pi} \phi(t) \left[\frac{\sin m\dfrac{t-x}{2r}}{m \sin \dfrac{t-x}{2r}}\right]^{2\kappa} dt.$$

Because of the periodicity of the functions involved, this is, except for the irrelevant constant factor $1/(2r)$, the same as

$$\int_{-r\pi}^{r\pi} \phi(t) \left[\frac{\sin m\dfrac{t-x}{2r}}{m \sin \dfrac{t-x}{2r}}\right]^{2\kappa} dt.$$

Let the interval of integration be broken up into r intervals, each of length 2π, and let these be reduced to a common interval by change of variable. In this way the last integral can be brought into the form

$$\int_{-r\pi}^{-(r-2)\pi} \phi(t) \sum_{j=0}^{r-1} \left[\frac{\sin m\left(\dfrac{t-x}{2r} + \dfrac{2j\pi}{2r}\right)}{m \sin\left(\dfrac{t-x}{2r} + \dfrac{2j\pi}{2r}\right)}\right]^{2\kappa} dt. \tag{41}$$

* The notation has been changed somewhat.

† Cf. B, pp. 347–348, where only the case $k = 1$ is taken up.

Now it is a familiar fact* that $[(\sin \frac{1}{2}mv)/(m \sin \frac{1}{2}v)]^2$ is a finite trigonometric sum in v, of order $m-1$, involving only cosines, and so, if this expression is raised to the κth power, there will be obtained a sum of the same character, of order $\kappa(m-1)$. That is, each term under the sign of summation in the integral just written down has the form

$$\sum_{i=0}^{\kappa(m-1)} B_i \cos i\left(\frac{t-x}{r}+\frac{2j\pi}{r}\right),$$

where the coefficients B_i are constants. On performing the summation with regard to j, a double sum is obtained which can be written thus:

$$(42) \qquad \sum_{i=0}^{\kappa(m-1)} \left\{ B_i \left[\cos i\frac{t-x}{r} \sum_{j=0}^{r-1} \cos \frac{2ij\pi}{r} - \sin i\frac{t-x}{r} \sum_{j=0}^{r-1} \sin \frac{2ij\pi}{r} \right] \right\}.$$

Since

$$\sum_{j=0}^{r-1} \cos \frac{2ij\pi}{r}$$

is zero unless i is divisible by r, and the corresponding sum of sines is always zero,† the expression (42) really involves no sines of multiples of $(t-x)/r$ at all, and the cosines of only such multiples of $(t-x)/r$ as are at the same time integral multiples of $t-x$. It has the same form as the right-hand side of (40), with the coefficients for which the second subscript is greater than n/r all equal to zero. As the integrand in (41), regarded as a function of t, is now seen to have the period 2π, the interval of integration may equally well be taken as that from $-\pi$ to π. To justify (39) and (40) it remains only to combine the terms corresponding to the several values of r.

The approximating property of $S_n(x)$, stated precisely, is this: If $\phi(x)$, of period 2π, has a $(k-1)$th derivative satisfying the condition (25), then, for all positive integral values of n and all values of x,

$$|\phi(x) - S_n(x)| \leqq \frac{K_k \mu}{n^k},$$

where K_k is a constant depending only on k.

Of course the number k enters into the definition of $S_n(x)$. If ϕ had a derivative of order $l-1$ satisfying a Lipschitz condition, $l > k$, it would not in general be true of this $S_n(x)$ that $|\phi(x) - S_n(x)|$ remains inferior to a constant multiple of $1/n^l$; to attain this degree of approximation, it would be necessary to define a new S_n by means of a new T_n. But it is readily seen, on following through the demonstration in A, that the function $T_n(t-x)$ formed for any particular value of k applies equally well for any *smaller*

* See Fejér, Mathematische Annalen, vol. 58 (1904), pp. 51–69; p. 53.

† See, e. g., Bôcher, *Introduction to the theory of Fourier's series*, Annals of Mathematics, ser. 2, vol. 7 (1906), pp. 81–152; p. 135.

value of k, and this is important for what follows. It will be well to give a summary of the facts that will be needed about the functions S_n and T_n in the form of a lemma.

LEMMA VI. *The positive integer k' being regarded as fixed, there exists for every positive integral value of n a cosine-sum $T_n(t-x)$, of the form* (40), *such that if $\phi(x)$ is any function of period 2π whose $(l-1)$th derivative, $l \leqq k'$, everywhere satisfies the condition*

$$|\phi^{(l-1)}(x_2) - \phi^{(l-1)}(x_1)| \leqq \mu |x_2 - x_1|,$$

then the corresponding $S_n(x)$, defined by (39), *satisfies for all values of x the relation*

$$|\phi(x) - S_n(x)| \leqq \frac{g\mu}{n^l},$$

where g depends only on k'.

For the purpose of obtaining a theorem on the approximate representation of a function which has a $(k-1)$th derivative satisfying a Lipschitz condition, by means of linear combinations of the functions $U_n(x)$, the number k' in the lemma is to be set equal to $k+1$, while l is to be given the values $k+1$ and k successively.

The former value is used in studying the behavior of the coefficients $A_{n\nu}$. Let $\cos \nu x$ be substituted for $\phi(x)$ in the lemma. It satisfies the relation

$$\left| \frac{d^k}{dx^k} \cos \nu x_2 - \frac{d^k}{dx^k} \cos \nu x_1 \right| \leqq \nu^{k+1} |x_2 - x_1|,$$

as is seen at once by applying the mean-value theorem to the kth derivative. Therefore

$$\left| \cos \nu x - \int_{-\pi}^{\pi} T_n(t-x) \cos \nu t \, dt \right| \leqq \frac{g\nu^{k+1}}{n^{k+1}}.$$

On the other hand, it follows from (40) that for $\nu \geqq 1$

$$\int_{-\pi}^{\pi} T_n(t-x) \cos \nu t \, dt = \pi A_{n\nu} \cos \nu x,$$

so that

$$|\cos \nu x - \pi A_{n\nu} \cos \nu x| \leqq \frac{g\nu^{k+1}}{n^{k+1}}$$

for all values of x, or, setting $x = 0$ in particular,

$$|1 - \pi A_{n\nu}| \leqq \frac{g\nu^{k+1}}{n^{k+1}}. \tag{43}$$

The same reasoning applied when $\nu = 0$ shows that

$$A_{n0} = \frac{1}{2\pi}$$

precisely.

It appears on substituting (40) in (39) that for any function $\phi(x)$

$$S_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi(t)\,dt + \sum_{\nu=1}^{n}\left\{A_{n\nu}\left[\cos\nu x\int_{-\pi}^{\pi}\phi(t)\cos\nu t\,dt + \sin\nu x\int_{-\pi}^{\pi}\phi(t)\sin\nu t\,dt\right]\right\}$$

$$= a_0 + \sum_{\nu=1}^{n} d_{n\nu}(a_\nu\cos\nu x + b_\nu\sin\nu x),$$

where a_ν, b_ν are the Fourier coefficients of ϕ, and the numbers $d_{n\nu} = \pi A_{n\nu}$ are independent of ϕ. That is, the functions $S_n(x)$ are formed by applying a particular method of summation to the Fourier's series for ϕ. Rewritten in the new notation, (43) states that

$$|1 - d_{n\nu}| \leqq \frac{g\nu^{k+1}}{n^{k+1}}. \tag{44}$$

For the sake of uniformity we will introduce a coefficient $d_{n0} = 1$.

We are now in a position to define the approximating function desired. Let it be assumed that $\phi(x)$ and $\lambda(x)$ satisfy the hypotheses of Theorem II, and let α_n denote as before the general coefficient in the Sturm-Liouville series for $\phi(x)$. The function to be used is the following:

$$\Sigma_n(x) = \sum_{\nu=0}^{n} d_{n\nu}\,\alpha_\nu\,U_\nu(x).$$

It will be shown that as n becomes infinite $\Sigma_n(x)$ converges uniformly to a function $\psi(x)$, which is then necessarily continuous, so rapidly that

$$|\psi(x) - \Sigma_n(x)|$$

does not exceed a constant multiple of $1/n^k$. When this has been done, it will remain to be proved that $\psi(x)$ and $\phi(x)$ are identical.

Suppose $\phi(x)$ defined for values of x outside of the interval $(0, \pi)$ so as to be an even function of period 2π. This function will have everywhere a $(k-1)$th derivative satisfying a Lipschitz condition of the form (25), so that Lemma VI is applicable with $l = k$. Since ϕ is even, $S_n(x)$ now has the form

$$S_n(x) = \sum_{\nu=0}^{n} d_{n\nu}\,a_\nu\cos\nu x,$$

where

$$a_\nu = \frac{2}{\pi}\int_0^{\pi}\phi(t)\cos\nu t\,dt, \qquad \nu \geqq 1,$$

with a corresponding formula for a_0.

Consider the difference

$$\Sigma_{n+p}(x) - \Sigma_n(x),$$

where p is any positive integer. It is equal to

$$S_{n+p}(x) - S_n(x) + \sum_{\nu=1}^{n} (d_{n+p,\nu} - d_{n\nu})[\alpha_\nu U_\nu(x) - a_\nu \cos \nu x] + \sum_{\nu=n+1}^{n+p} d_{n+p,\nu}[\alpha_\nu U_\nu(x) - a_\nu \cos \nu x]. \tag{45}$$

By Lemma VI,

$$|S_{n+p}(x) - S_n(x)| = |[\phi(x) - S_n(x)] - [\phi(x) - S_{n+p}(x)]| \leqq \frac{2g\mu}{n^k}.$$

From (44),

$$|d_{n+p,\nu} - d_{n\nu}| \leqq \frac{2g\nu^{k+1}}{n^{k+1}},$$

and from (27), the correctness of which was established under the hypotheses that we are using now,

$$|\alpha_\nu U_\nu(x) - a_\nu \cos \nu x| \leqq \frac{\beta}{\nu^{k+1}}, \tag{46}$$

where β is a constant. Hence the absolute value of each term under the first sign of summation in (45) is less than or equal to $2g\beta/n^{k+1}$, and the absolute value of the sum of n terms does not exceed $2g\beta/n^k$. As for the other sum, it follows from (44) that

$$|d_{n+p,\nu}| \leqq 1 + g,$$

for all values of n, p, and ν, while (46) is still satisfied, and so

$$\left|\sum_{\nu=n+1}^{n+p} d_{n+p,\nu}[\alpha_\nu U_\nu(x) - a_\nu \cos \nu x]\right| \leqq \beta(1+g) \sum_{\nu=n+1}^{n+p} \frac{1}{\nu^{k+1}} < \beta(1+g) \int_n^\infty \frac{dt}{t^{k+1}} = \frac{\beta(1+g)}{kn^k}.$$

Combining these facts, we see that the absolute value of $\Sigma_{n+p}(x) - \Sigma_n(x)$ does not exceed $1/n^k$ multiplied by a quantity independent of x, n, and p. Consequently $\Sigma_n(x)$ does uniformly approach a continuous limiting function $\psi(x)$ as n becomes infinite, and

$$\psi(x) = \Sigma_n(x) + O\left(\frac{1}{n^k}\right)$$

uniformly for all values of x in the interval from 0 to π.

There would be no difficulty in showing that if $\phi^{(k-1)}(0) = 0$, the absolute value of $\psi(x) - \Sigma_n(x)$ does not exceed $c\mu/n^k$, where c is independent of ϕ; but as in other cases of the same sort we will not go into detail on this point.

It is still to be proved that the limiting function $\psi(x)$ is identical with $\phi(x)$. It might be interesting to know whether it is true that a convergent series will always be summable by means of the factors $d_{n\nu}$, and that the value so obtained will always be equal to the sum of the series. We shall leave this general question aside, and confine our attention to the special problem in hand, making use of the fact that $s_n(x)$, $S_n(x)$, and $\sigma_n(x)$ all converge to the value $\phi(x)$. The proof is as follows:

Let x be any number in the interval $0 \leqq x \leqq \pi$.

Let ϵ be any positive quantity.

Let $S_n^q(x)$ denote the sum of the first $q+1$ terms of $S_n(x)$, and $\Sigma_n^q(x)$ the sum of the first $q+1$ terms of $\Sigma_n(x)$, when $n \geqq q$.

Two integers n and q are to be chosen subject to seven conditions; the possibility of fulfilling these conditions will be made clear after the conditions themselves have been written down.

Let q be a number such that

$$|\phi(x) - s_q(x)| < \epsilon/7, \tag{47}$$

$$|\sigma_q(x) - \phi(x)| < \epsilon/7, \tag{48}$$

$$|[\Sigma_n(x) - \Sigma_n^q(x)] - [S_n(x) - S_n^q(x)]| < \epsilon/7 \tag{49}$$

for every value of $n > q$. Let q, once chosen, be held fast, and let a number $n > q$ be chosen so that

$$|s_q(x) - S_n^q(x)| < \epsilon/7, \tag{50}$$

$$|\Sigma_n^q(x) - \sigma_q(x)| < \epsilon/7, \tag{51}$$

$$|S_n(x) - \phi(x)| < \epsilon/7, \tag{52}$$

$$|\psi(x) - \Sigma_n(x)| < \epsilon/7. \tag{53}$$

That (47), (48), (52), and (53) can be satisfied, is an immediate consequence of the convergence of the respective sums involved. In (50),

$$s_q(x) - S_n^q(x) = \sum_{\nu=1}^{q} (1 - d_{n\nu})\, a_\nu \cos \nu x,$$

and for a fixed value of q the right-hand member involves n only in the differences $1 - d_{n\nu}$, and approaches zero as n becomes infinite. Similar reasoning applies to (51). In the remaining condition (49),

$$\Sigma_n(x) - \Sigma_n^q(x) = \sum_{\nu=q+1}^{n} d_{n\nu}\, \alpha_\nu\, U_\nu(x),$$

$$S_n(x) - S_n^q(x) = \sum_{\nu=q+1}^{n} d_{n\nu}\, a_\nu \cos \nu x,$$

and the difference of these expressions is, except as to notation, the second sum in (45), and is inferior in absolute value to a quantity which is independent of n and approaches zero as q becomes infinite. Hence (49) also can be fulfilled. Of course it will be essential that all seven conditions be satisfied simultaneously, but as each by itself is satisfied for *all* values of q or of n from some point on, there will be no difficulty in finding a single value of q for the first three, and then a single value of n for the last four.

With the justification of the seven inequalities, the proof is practically complete. It is found upon adding that the sum of the quantities inclosed in bars on the left-hand sides is simply $\psi(x) - \phi(x)$, and as the absolute value of this sum can not exceed the sum of the absolute values, it follows that

$$|\phi(x) - \psi(x)| < \epsilon$$

for suitable values of q and n. But ϕ and ψ are independent of q and n, and as ϵ is arbitrarily small, it must be that $\psi(x) = \phi(x)$ exactly. As this has been proved for a value of x which is any value in $(0, \pi)$, the equation is an identity. The conclusion is as follows:

THEOREM IV. *If $\phi(x)$ and $\lambda(x)$ satisfy the hypotheses of Theorem II, there exists for every positive integral value of n a linear combination $\Sigma_n(x)$ of the functions $U_\nu(x)$, $\nu = 0, 1, \cdots, n$, with constant coefficients, such that*

$$\phi(x) = \Sigma_n(x) + O\left(\frac{1}{n^k}\right)$$

uniformly throughout the interval $0 \leqq x \leqq \pi$.

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SINGULAR INTEGRAL EQUATIONS OF THE VOLTERRA TYPE*

BY

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1. Introduction. It has been shown by Evans† that the equation

$$\phi(x) = f(x) + \int_x^\infty K(x,t)\phi(t)\,dt \tag{1}$$

has a unique finite and integrable solution provided that, for all values of $x \geqq a$ and of $t \geqq x$, (*a*) the functions $f(x)$ and $K(x,t)$ are bounded and continuous; (*b*) the integral

$$\int_x^\infty |K(x,t)|\,dt$$

exists; (*c*) a constant b can be found such that

$$\int_x^\infty |K(x,t)|\,dt \leqq N < 1 \qquad (x \geqq b).$$

Equations of the form (1) sometimes arise,‡ however, for which the conditions of Evans's theorem are not satisfied. Various cases in which this is true are considered in the present paper. In each instance an attempt is made not merely to prove the existence of a continuous solution, but also to determine its behavior for large values of x.

The independent variables x and t will be restricted to real values. Functions of the single variable x will be considered for values of x lying either on the range

I $\qquad x \geqq a$

or on the range

I_b $\qquad a \leqq x \leqq b,$

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† Atti della Reale Accademia dei Lincei, ser. 5, vol. 20 (1911), pp. 656–662. Cf. also: Ibid., vol. 20 (1911), pp. 7–11.

‡ For instance, in the theory of linear differential equations. Cf. Dini, Annali di Matematica, ser. 3, vol. 2 (1899), pp. 297–324; ibid., vol. 3 (1900), pp. 125–183; also papers by the present writer: Annals of Mathematics, ser. 2, vol. 15 (1914), pp. 145–156, and American Journal of Mathematics, vol. 36 (1914), pp. 151–166.

where b is arbitrary; functions of the two variables x, t will be considered for values of x and t in the domain

$$T \qquad x \geqq a, \qquad t \geqq x.$$

For brevity we shall speak of these three regions as the interval I, the interval I_b, and the region T respectively.

The functions $f(x)$ and $K(x,t)$ are assumed throughout to be continuous in I and in T respectively.

2. Theorem 1. *If the integral*

$$\int_x^\infty |K(x,t)f(t)|\,dt$$

converges in I, converges uniformly in I_b, and has in I the property that

$$\int_x^\infty |K(x,t)f(t)|\,dt < \theta|f(x)| \qquad (0<\theta<1), \tag{2}$$

where θ is a constant independent of x, then equation (1) *has in I a continuous solution $\phi(x)$ expressible in the form*

$$\phi(x) = f(x)\rho(x), \tag{3}$$

where $\rho(x)$ is a continuous function such that

$$|\rho(x)| < 1/(1-\theta).$$

The method of iteration,* when applied to the equation (1), leads to consideration of the series

$$\phi(x) = \sum_{n=0}^{\infty} u_n(x), \tag{4}$$

where

$$u_0(x) = f(x),$$

$$u_n(x) = \int_x^\infty \int_t^\infty \cdots \int_{t_{n-3}}^\infty \int_{t_{n-2}}^\infty K(x,t)K(t,t_1) \cdots K(t_{n-3},t_{n-2})K(t_{n-2},t_{n-1})f(t_{n-1})\,dt_{n-1}\,dt_{n-2}\cdots dt_1\,dt$$

$$= \int_x^\infty K(x,t)u_{n-1}(t)\,dt \qquad (n=1,2,\cdots).$$

We shall first show by mathematical induction that, in I, $u_n(x)$ is continuous and satisfies the inequality

$$|u_n(x)| < \theta^n|f(x)| \qquad (n=1,2,\cdots). \tag{5}$$

* Cf., for example, Bôcher, *An Introduction to the Study of Integral Equations* (1909), p. 14.

Assume that $u_{n-1}(x)$ is continuous in I, and that

$$|u_{n-1}(x)| \leqq \theta^{n-1}|f(x)|.$$

Then $u_n(x)$ is continuous in I and satisfies (5). For, since the integral

$$\int_x^\infty |K(x,t)f(t)|dt$$

converges uniformly in I_b, we can find for every value of $\epsilon > 0$ a value of μ independent of x, such that

$$\int_\mu^{\mu'} |K(x,t)f(t)|dt < \epsilon, \tag{6}$$

where μ' has any value greater than μ, and x lies in I_b. Now

$$\left|\int_\mu^{\mu'} K(x,t)u_{n-1}(t)\,dt\right| \leqq \int_\mu^{\mu'} |K(x,t)u_{n-1}(t)|dt$$
$$\leqq \theta^{n-1}\int_\mu^{\mu'} |K(x,t)f(t)|dt < \epsilon,$$

by (5) and (6). But this last relation establishes the uniform convergence of $u_n(x)$ in I_b, and hence its continuity at all points in I, since an interval I_b can always be found so as to include any preassigned point in I. Further,

$$|u_n(x)| \leqq \int_x^\infty |K(x,t)u_{n-1}(t)|dt \leqq \theta^{n-1}\int_x^\infty |K(x,t)f(t)|dt < \theta^n|f(x)|,$$

by (2).

To complete the proof by induction we need only note that our assumption regarding $u_{n-1}(x)$ is justified, by hypothesis, for $n = 1$.

As a consequence of (5), we have

$$\sum_{n=0}^{\infty} |u_n(x)| < \frac{1}{1-\theta}|f(x)|, \tag{7}$$

so that the series $\sum_{n=0}^{n=\infty} u_n(x)$ is absolutely convergent in I, and $\phi(x)$ as given by (4) is defined in that interval. Further, the same series is uniformly convergent in I_b, by Weierstrass's test, since

$$|u_n(x)| < N\theta^n,$$

where N is the maximum value of $|f(x)|$ in I_b. The function $\phi(x)$ is therefore the sum of a uniformly convergent series of continuous functions, and is itself continuous.

Let us now multiply the series for $\phi(t)$ by $K(x,t)$ and integrate term by term from x to ∞. That this is allowable appears from the following theorem:*

* Bromwich, *Infinite Series* (1908), p. 453.

If the series $\sum_{n=0}^{n=\infty} f_n(x)$ *converges uniformly in any fixed interval* $a \leqq x \leqq b$, *where* b *is arbitrary, and if* $F(x)$ *is continuous for all finite values of* x, *then*

$$\int_a^\infty F(x)\Big[\sum_{n=0}^{\infty} f_n(x)\Big]\,dx = \sum_{n=0}^{\infty}\int_a^\infty F(x) f_n(x)\,dx,$$

provided that the integral

$$\Phi(x) = \int_a^\infty |F(x)| \sum_{n=0}^{\infty} |f_n(x)|\,dx,$$

converges.

For, if we take $a = x$, $x = t$, $F(t) = K(x, t)$, $f_n(t) = u_n(t)$, it is only necessary to show that the integral $\Phi(x)$ converges. But this follows at once from (7) and (2).

Upon performing the term-by-term integration mentioned above, we obtain the equation

$$\int_x^\infty K(x, t)\phi(t)\,dt = \sum_{n=1}^{\infty} u_n(x) = \phi(x) - f(x),$$

which shows that $\phi(x)$ as given by (4) is a solution of (1).

From (7) it appears at once that we may write $\phi(x)$ in the form (3). The function $f(x)$ cannot vanish in I, by (2), so that $\rho(x)$ is continuous. This completes the proof of the theorem.

That the solution thus obtained is not necessarily the only continuous solution* appears from the following example. The hypotheses of the theorem are evidently satisfied by the equation

$$\phi(x) = e^x + \tfrac{1}{2}e^{2x}\int_x^\infty e^{-2t}\phi(t)\,dt,$$

since

$$\int_x^\infty |K(x, t)f(t)|\,dt = \tfrac{1}{2}e^{2x}\int_x^\infty e^{-t}\,dt = \tfrac{1}{2}e^x.$$

The equation evidently has the infinity of solutions

$$\phi(x) = 2e^x + ce^{3x/2},$$

where c is an arbitrary constant, the solution given by the theorem being obtained by taking $c = 0$.

Similar examples can be constructed for each of the cases still to be treated.

In case the hypotheses of the theorem are not satisfied for all values of

* It may be remarked that in the case treated by Evans (loc. cit.) there may likewise be more than one continuous solution. The equation

$$\phi(x) = 1 + \tfrac{1}{2}e^x\int_x^\infty e^{-t}\phi(t)\,dt,$$

which evidently satisfies the conditions of Evans's theorem, has the family of continuous solutions

$$\phi(x) = 2 + ce^{x/2},$$

where c is an arbitrary constant.

x in I, but only for sufficiently large values, say $x \geqq a' > a$, we may proceed by "prolongement"* to obtain a continuous solution valid throughout I. Write the equation (1) in the form

$$\phi(x) = f_1(x) + \int_x^{a'} K(x,t)\phi(t)\,dt, \tag{8}$$

where

$$f_1(x) = f(x) + \int_{a'}^{\infty} K(x,t)\phi(t)\,dt.$$

Our theorem may be used to determine $\phi(x)$ for $x \geqq a'$, after which $f_1(x)$ becomes a known function, and (8) reduces to a *regular* Volterra equation whose solution in the interval $a \leqq x \leqq a'$ may be found by the usual methods.

A similar remark will apply to each of the later theorems.

3. Theorem 2. *If there exists a continuous positive real function $h(x)$ such that the integrals*

$$\int_x^{\infty} |K(x,t)f(t)|\,dt, \qquad \int_x^{\infty} |K(x,t)|h(t)\,dt$$

converge in I, converge uniformly in I_b, and satisfy in I the inequalities

$$\int_x^{\infty} |K(x,t)f(t)|\,dt < h(x), \tag{9}$$

$$\int_x^{\infty} |K(x,t)|h(t)\,dt < \theta h(x) \qquad (0 < \theta < 1), \tag{10}$$

where θ is a constant independent of x, then equation (1) has in I a continuous solution $\phi(x)$ expressible in the form

$$\phi(x) = f(x) + h(x)\rho(x), \tag{11}$$

where $\rho(x)$ is a continuous function such that

$$|\rho(x)| < 1/(1-\theta).$$

COROLLARY. *If the integrals*

$$\int_x^{\infty} |K(x,t)f(t)|\,dt, \qquad \int_x^{\infty} |K(x,t)|\,dt$$

converge in I, converge uniformly in I_b, and satisfy in I the inequalities

$$\int_x^{\infty} |K(x,t)f(t)|\,dt < N, \qquad \int_x^{\infty} |K(x,t)|\,dt < \theta,$$

where N is a constant, then equation (1) has in I a continuous solution $\phi(x)$ expressible in the form

$$\phi(x) = f(x) + \rho(x),$$

where

$$|\rho(x)| < N/(1-\theta).$$

* Cf. Evans, loc. cit., p. 662.

To prove the theorem, consider again the series (4). We note first that $u_n(x)$ is continuous in I and satisfies the inequality

$$|u_n(x)| < \theta^{n-1} h(x) \qquad (n = 1, 2, \cdots). \tag{12}$$

For, if we assume that $u_{n-1}(x)$ has these properties, the same will be true for $u_n(x)$, by argument similar to that of § 2. But the assumption holds for $u_1(x)$, by hypothesis.

It follows from (12) that the series (4) converges absolutely in I and uniformly in I_b. By making obvious modifications in the argument of § 2, we see that $\phi(x)$ as given by (4) is continuous in I, and satisfies equation (1) in the same interval.

By (12), we have

$$\sum_{n=1}^{\infty} |u_n(x)| < \frac{1}{1-\theta} h(x),$$

so that

$$|\phi(x) - f(x)| < \frac{1}{1-\theta} h(x).$$

The form (11) for $\phi(x)$ results at once.

The corollary is merely that case of the theorem in which we may take

$$h(x) = N.$$

4. Theorem 3. *If there exists a continuous positive real function $\eta(x)$ for which the integral*

$$\int_x^{\infty} \eta(t)\, dt$$

exists, and such that, in T,

$$|K(x,t)f(t)| < |f(x)|\eta(t), \tag{13}$$

then equation (1) *has in I a continuous solution $\phi(x)$ expressible in the form*

$$\phi(x) = f(x)[1 + \epsilon(x)],$$

where $\epsilon(x)$ is continuous and has the property that

$$\lim_{x=\infty} \epsilon(x) = 0.$$

We may evidently assume without loss of generality that

$$\int_x^{\infty} \eta(t)\, dt < \theta \qquad (0 < \theta < 1). \tag{14}$$

For, if this is not true throughout I, it will be true in the interval $x \geqq a'$, where a' is sufficiently large. After establishing the existence of a continuous solution of (1) in this latter interval, we may by prolongement* extend it to all values of x in I.

* Cf. § 2.

It results from (13) and (14) that

$$\int_x^\infty |K(x,t)f(t)|dt < |f(x)| \int_x^\infty \eta(t)\,dt < \theta|f(x)| \qquad (0<\theta<1).$$

Further, the integral

$$\int_x^\infty |K(x,t)f(t)|dt$$

converges uniformly in I_b, by the following (Weierstrass's) test.†

Suppose that for all values of x in the interval $a \leqq x \leqq b$ the function $F(x,t)$ satisfies the condition

$$|F(x,t)| < M(t),$$

where $M(t)$ is a positive function, independent of x. Then if the integral

$$\int_x^\infty M(t)\,dt$$

converges, the integral

$$\int_x^\infty F(x,t)\,dt$$

is absolutely and uniformly convergent for all values of x in the interval $a \leqq x \leqq b$.

Let us write

$$\int_x^\infty |K(x,t)f(t)|dt = \int_x^b |K(x,t)f(t)|dt + \int_b^\infty |K(x,t)f(t)|dt.$$

The test applies at once to the integral last written if we take

$$m = b, \qquad F(x,t) = |K(x,t)f(t)|, \qquad M(t) = N\eta(t),$$

where N is the maximum value of $|f(x)|$ in I_b.

Thus the conditions of Theorem 1 are satisfied in the present case, and by that theorem there exists a continuous solution of (1) of the form

$$\phi(x) = f(x)\rho(x).$$

If we put

$$\rho(x) = 1 + \epsilon(x),$$

it appears by (4) that

$$f(x)\epsilon(x) = \sum_{n=1}^\infty u_n(x),$$

and by (7) we have

$$|f(x)\epsilon(x)| \leqq \sum_{n=1}^\infty |u_n(x)| < \frac{\theta}{1-\theta}|f(x)|,$$

or

$$|\epsilon(x)| < \theta/(1-\theta).$$

† Cf., for instance, Bromwich, *Infinite Series*, p. 434.

Now θ may be taken arbitrarily small if x be taken sufficiently large, so that

$$\lim_{x=\infty} \epsilon(x) = 0.$$

This completes the proof.

5. Theorem 4. *If there exist functions $h(x)$ and $\eta(x)$* such that the inequalities*

$$|K(x,t)f(t)| < h(x)\eta(t), \tag{15}$$

$$|K(x,t)|h(t) < h(x)\eta(t) \tag{16}$$

are satisfied in T, then equation (1) has in I a continuous solution $\phi(x)$ expressible in the form

$$\phi(x) = f(x) + h(x)\epsilon(x),$$

where $\epsilon(x)$ is continuous and has the property that

$$\lim_{x=\infty} \epsilon(x) = 0.$$

COROLLARY. *If there exists a function $\eta(x)$ such that the inequalities*

$$|K(x,t)f(t)| < \eta(t), \qquad |K(x,t)| < \eta(t)$$

are satisfied in T, then equation (1) has in I a continuous solution $\phi(x)$ expressible in the form

$$\phi(x) = f(x) + \epsilon(x),$$

where

$$\lim_{x=\infty} \epsilon(x) = 0.$$

From (15) and (16) it appears that (9) and (10) are satisfied. The integrals

$$\int_x^\infty |K(x,t)f(t)|dt \quad \text{and} \quad \int_x^\infty |K(x,t)|h(t)\,dt$$

are readily seen to satisfy the other conditions of Theorem 2. Hence equation (1) has a continuous solution

$$\phi(x) = f(x) + h(x)\epsilon(x),$$

where

$$\epsilon(x) = \rho(x) = \frac{u_1(x)}{h(x)} + \frac{1}{h(x)}\sum_{n=2}^{\infty} u_n(x).$$

Now by (15)

$$\frac{|u_1(x)|}{h(x)} < \theta,$$

and by (12)

$$\frac{1}{h(x)}\sum_{n=2}^{\infty} |u_n(x)| < \frac{\theta}{1-\theta},$$

* The functions $h(x)$ and $\eta(x)$ have the same properties as in Theorem 2 and Theorem 3 respectively.

so that

$$|\epsilon(x)| < \theta + \theta/(1-\theta).$$

The desired result follows as in § 4.

To prove the corollary, take $h(x) = 1$.

6. Application. The foregoing results may be used in a variety of ways to discuss particular equations of the form (1). The following may be mentioned as a typical application:

Let $p(x)$, $q(x)$, and $r(x)$ be polynomials in x such that for large values of x

$$R[p(x)] < R[q(x)] < R[r(x)],$$

where $R[x]$ means the real part of x. If for sufficiently large values of x, and for values of $t \geqq x$, we may write, in Poincaré's sense,*

$$f(x) \sim e^{q(x)}\, x^{\alpha}\left(a_0 + \frac{a_1}{x} + \cdots\right),$$

$$K(x,t) \sim e^{p(x)}\, x^{\beta}\, e^{-r(t)}\, t^{\gamma}\left(b_{0,0} + \frac{b_{1,0}}{x} + \frac{b_{0,1}}{t} + \frac{b_{2,0}}{x^2} + \cdots\right),$$

where α, β, γ, $a_0, \cdots, b_{0,0}, \cdots$ are determinate constants, then for the same values of x the equation (1) has a continuous solution $\phi(x)$ such that we may write

$$\phi(x) \sim f(x).$$

To prove this statement, we note first that x may be taken so large that

$$|K(x,t)| < N_1 |e^{p(x)}\, x^{\beta}|\,|e^{-r(t)}\, t^{\gamma}|, \qquad |f(x)| < N_2 |e^{q(x)}\, x^{\alpha}|,$$

where N_1 and N_2 are certain constants. Thus

$$|K(x,t) f(t)| < N_1 N_2 |e^{p(x)}\, x^{\beta}\, e^{q(t)-r(t)}\, t^{\alpha+\gamma}|. \tag{17}$$

Let us now apply Theorem 4, taking

$$h(x) = |e^{p(x)}\, x^{\beta}|, \qquad \eta(x) = N|e^{q(x)-r(x)}|\, x^{k},$$

where N is the larger of the quantities N_1 and $N_1 N_2$, and k is the larger of the quantities

$$R[\alpha+\beta], \qquad R[\alpha+\gamma].$$

Since

$$R[q(x)] < R[r(x)],$$

the integral

$$\int_x^{\infty} \eta(t)\, dt$$

*Acta Mathematica, vol. 8 (1886), p. 296.

exists. We have directly, by (17), the inequality

$$|K(x,t)f(t)| < h(x)\eta(t).$$

Also

$$|K(x,t)|h(t) < N_1|e^{p(x)}\,x^{\beta}\,e^{p(t)-r(t)}\,t^{\beta+\gamma}| < h(x)\eta(t),$$

since

$$R[p(x)] < R[q(x)]. \tag{18}$$

Thus the hypotheses of Theorem 4 are satisfied, and equation (1) has a continuous solution of the form

$$\begin{aligned}\phi(x) &= f(x) + h(x)\epsilon(x)\\ &= e^{q(x)}\,x^{\alpha}\left[a_0 + \frac{a_1}{x} + \cdots + \frac{a_n + \omega_n(x)}{x^n}\right] + |e^{p(x)}\,x^{\beta}|\epsilon(x),\end{aligned}$$

where n is an arbitrary positive integer and

$$\lim_{x=\infty}\omega_n(x) = 0, \qquad \lim_{x=\infty}\epsilon(x) = 0.$$

Put

$$\bar{\epsilon}(x) = \frac{\epsilon(x)|e^{p(x)}\,x^{\beta}|}{e^{p(x)}\,x^{\beta}},$$

$$\bar{\omega}_n(x) = \omega_n(x) + e^{p(x)-q(x)}\,x^{\beta-\alpha+n}\,\bar{\epsilon}(x).$$

Then we have

$$\phi(x) = e^{q(x)}\,x^{\alpha}\left[a_0 + \frac{a_1}{x} + \cdots + \frac{a_n + \bar{\omega}_n(x)}{x^n}\right].$$

It appears by (18) that

$$\lim_{x=\infty}\bar{\omega}_n(x) = 0,$$

whence we may write

$$\phi(x) \sim e^{q(x)}\,x^{\alpha}\left(a_0 + \frac{a_1}{x} + \cdots\right).$$

Thus $\phi(x)$ has an asymptotic expansion coinciding with that of $f(x)$, which was to be proved.

It is believed that the present theorems may prove useful in the study of certain problems in the theory of linear differential equations. (The writer hopes to consider applications of this type in a later paper.)

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ON THE REDUCTION OF INTEGRO-DIFFERENTIAL EQUATIONS*

BY

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Certain integro-differential equat ons can be completely treated by first resolving an integral equation, and second, integrating a differential equation. In so far, such equations do not constitute a new problem in analysis, because they are reducible to equations of simpler types. From this point of view Volterra discusses the equation

$$\nabla^2 V(x, y, z|t) + \int_{t_0}^{t} f(t, \tau)\, \nabla^2 V(x, y, z|\tau)\, d\tau = F(x, y, z, t)$$

which is satisfied by the potential V of an electric field in an isotropic medium, in the so-called "static case" of hysteresis.† By solving this equation, regarded as an integral equation, for $\nabla^2 V$, we are led to the differential equation of Poisson to determine V.

We may however have the case that an integro-differential equation whose solution is subject to certain boundary conditions is reducible to one or more problems in equations of more elementary character only when account is taken of these conditions. This is the point of view to be developed in the present paper. It is a point of view which can be extended to more general types of functional equations.

We shall treat, in the following pages, the reduction of boundary value problems in the case of the integro-differential equation of parabolic type, and quite briefly that of hyperbolic type, to problems in linear differential and integral equations separately. It is worth while perhaps to specially mention § 2, in which the generalization of the integro-differential equation of the second order to an equation of the first order involving integration

* Presented to the Society, December, 1912, and December, 1913. Since the first writing, the paper has been entirely revised, partly to include new material, and partly to take advantage of the material of Prof. W. A. Hurwitz's thesis, *Randwertaufgaben bei Systemen von linearen partiellen Differentialgleichungen erster Ordnung*, Diss. Göttingen, 1910, of which the author was ignorant at the time of the first writing.

† V. Volterra, *Lectures delivered at the celebration of the 20th anniversary of the foundation of Clark University* (Sept., 1911), p. 75; *Sulle equazioni integro differenziali*, Rendiconti della R. Accademia dei Lincei, vol. 18 (1909), p. 174; *Sulle equazioni integro-differenziali della teoria dell'elasticità*, ibid., vol. 18 (1909), p. 297.

around an arbitrary path demands a corresponding restatement and treatment of the non-homogeneous linear partial differential equation of parabolic type.*

1. The Integro-Differential Equation of Parabolic Type

1. The region and its corresponding equation. For the parabolic *differential* equation the earliest boundary value problems referred to an open rectangle as the boundary. Afterwards the results were extended to curves of a more general nature.

In connection with the integro-differential equation

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = \int_{t_0}^{t} A(t,\tau)\,\frac{\partial u(x,\tau)}{\partial \tau}\,d\tau$$

there is a boundary value problem based upon the open rectangle: if under proper conditions a chain of values is given along the three sides of this rectangle, the given equation then uniquely determines them along the fourth side. In fact, for all the partial integro-differential equations heretofore considered, the boundaries have been of this sort. And if we wish to extend the results to a more general type of region, it is also opportune for us, as it turns out, to consider a slightly more general type of integro-differential equation.

We shall take as the open boundary the curves $t = t_0$ on the left, and the curves $x = \xi_2(t)$ above, and $x = \xi_1(t)$ below. We shall assume that $\xi_1(t)$ and $\xi_2(t)$ and their first and second derivatives remain single valued, finite and continuous; moreover that $\xi_2(t) > \xi_1(t)$ for any value of $t \geqq t_0$. The derivatives $\xi_1'(t)$ and $\xi_2'(t)$ are to vanish only a finite number of times in the interval $t_0 \leqq t \leqq t_1$. For simplicity we shall impose also the unnecessary restriction that $x = \xi_1(t)$ shall have no maximum and $x = \xi_2(t)$ no minimum, in the interval $t_0 < t < t_1$.†

We shall adopt Professor Hurwitz's convenient notation ${}_a\Omega_b$ for the space bounded by these curves and by the straight lines $t = a$ on the left and $t = b$ on the right.

Along the open boundary is to be given the continuous chain of values

$$(\alpha)\quad F(x, T_x) = \left\{\begin{array}{lll} f(x) & \text{when} & t = t_0 \qquad [\xi_1(t_0) \leqq x \leqq \xi_2(t_0)] \\ \phi_1(t) & \text{when} & x = \xi_1(t) \\ \phi_2(t) & \text{when} & x = \xi_2(t) \end{array}\right\} \qquad (t_0 \leqq t \leqq t_1)$$

* For an extensive treatment of the differential equation of parabolic type see Hurwitz, loc. cit., on which our § 1 is based, and also for a treatment that demands different conditions E. E. Levi, *Sul l'equazione del calore*, Annali di Matematica, vol. 14 (1908), p. 187.

† This restriction may be removed by an extension of the present method of proof, or by a continuous prolongation of the equation and its solution.

where the functions $f(x)$, $\phi_1(t)$, and $\phi_2(t)$ are supposed to have continuous first derivatives in the range under consideration, and T_x is used to denote the t-coördinate of the left-hand point on the boundary whose other coördinate is x. When it is necessary to use the t-coördinate of the right-hand boundary point whose other coördinate is x, we shall denote it by t_x, t_{1x}, etc., according as the right-hand limit of the region is formed by $t = t$, $t = t_1$, etc.

In connection with this region and these boundary values, we shall investigate the equation

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = \int_{T_x}^{t} A(x,t,\tau)\frac{\partial u(x,\tau)}{\partial \tau}\,d\tau. \tag{1}$$

We shall say that a function $u(x,t)$ is regular with respect to this equation if within the region and on its boundary u and $\partial u/\partial x$ remain finite and continuous, and within the region $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ exist except on a regularly distributed set of curves,* and are integrable linearly in the Riemann sense.

2. The theorem of reducibility for equation (1). In equation (1) let $A(x,t,\tau)$ and its first partial derivatives with respect to the three arguments be finite and continuous functions within and on the boundary of the region, and inside that boundary let $\partial A(x,t,\tau)/\partial\tau = A_1(x,t,\tau)$ have a derivative in regard to x that remains finite and continuous, except on a regularly distributed set of curves.

Under these conditions, there is one and only one regular solution $u(x,t)$ of the equation (1), which joins on continuously to the boundary values (α). It can be reduced to the form

$$u(x_1,t_1) = u'(x_1,t_1) - \int\int_{t_0\Omega t_1} k(x_1,t_1|x,t)\,u'(x,t)\,dx\,dt, \tag{2}$$

in which $u'(x,t)$ is the regular solution of the differential equation

$$\frac{\partial u'(x,t)}{\partial t} - \frac{\partial^2 u'(x,t)}{\partial x^2} = -A(x,t,T_x)F(x,T_x) \tag{3}$$

which takes on the boundary values (α), and $k(x_1,t_1|x,t)$ is the function associated to

$$K(x_1,t_1|x,t) = \frac{1}{2\sqrt{\pi}}\left\{ g(x,t|x_1,t_1)A(x,t,t) - \int_t^{t_{1x}} A_1(x,\tau,t)\,g(x,\tau|x_1,t_1)\,d\tau \right\} \tag{4}$$

by means of Volterra's relation

* See M. Bôcher, *Introduction to Integral Equations* (1909), p. 3.

$$k(x_1,t_1|x,t)+K(x_1,t_1|x,t)=\int\int_{t\Omega_{t_1}} K(x_1,t_1|\xi,\tau)\,k(\xi,\tau|x,t)\,d\xi\,d\tau$$

$$=\int\int_{t\Omega_{t_1}} k(x_1,t_1|\xi,\tau)\,K(\xi,\tau|x,t)\,d\xi\,d\tau. \tag{5}$$

By $g(x,t|x_1,t_1)$ is denoted the Green's function for the equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}.^*$$

It shou'd be noticed that $u'(x,t)$ depends on the boundary conditions, but that $K(x_1,t_1|x,t)$ is independent of them, and is determined merely by the coefficients of the equation and the nature of the region. We shall now give a proof of this theorem.

3. The equivalence of the integral and the integro-differential equations. By means of an integration by parts, the equation (1) may be rewritten in the form

$$\frac{\partial u(x,t)}{\partial t}-\frac{\partial^2 u(x,t)}{\partial x^2}=A(x,t,t)\,u(x,t)-A(x,t,T_x)\,F(x,T_x)$$

$$-\int_{T_x}^{t} A_1(x,t,\tau)\,u(x,\tau)\,d\tau. \tag{6}$$

We have the theorem:† *If $\phi(x,y)$ is continuous in the given region, and if in the neighborhood of a point (x_1,y_1) within that region $\partial\phi/\partial x$ exists and is continuous, then the function*

$$\bar{u}(x_1,t_1)=\frac{1}{2\sqrt{\pi}}\int\int_{t_0\Omega_{t_1}} g(x,t|x_1,t_1)\,\phi(x,t)\,dx\,d$$

is regular in the neighborhood of the point (x_1,y_1) and in that neighborhood satisfies the differential equation

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=\phi(x,t).$$

Moreover it vanishes on the open boundary of the region Ω.

If we examine the function $A(x,t,T_x)F(x,T_x)$ we notice that its

* We may write

$$g(x,t|x_1,t_1)=h_{\alpha/2}+g',$$

where g' is regular and

$$h_{\alpha\beta}(x,t|x_1,t_1)=\frac{(x_1-x)^\alpha}{(t_1-t)^\beta}\,e^{-\frac{(x_1-x)^2}{4(t_1-t)}}$$

(see for instance loc. cit. Levi, § 7).

† This is a slight but obvious generalization of Hurwitz's theorem VIII *b* (Hurwitz, loc. cit., p. 85), to which the same proof applies.

derivative in regard to x exists and is continuous except on a finite number of lines parallel to the t-axis, namely, those lines which pass through points where $\xi_1'(t)$ or $\xi_2'(t)$ vanishes, and the two lines $x = \xi_1(t_0)$ and $x = \xi_2(t_0)$. Hence we know that the regular solution $u'(x, t)$ of (3), which satisfies the given conditions (α), exists and is uniquely determined. If we write then $u(x, t) = u'(x, t) + v(x, t)$ we have the following integro-differential equation to determine $v(x, t)$;

$$\begin{aligned}(7)\qquad \frac{\partial v(x,t)}{\partial t} - \frac{\partial^2 v(x,t)}{\partial x^2} &= A(x,t,t)v(x,t) - \int_{T_x}^{t} A_1(x,t,\tau)v(x,\tau)\,d\tau \\ &\quad + A(x,t,t)u'(x,t) - \int_{T_x}^{t} A_1(x,t,\tau)u'(x,\tau)\,d\tau,\end{aligned}$$

with the condition that it be regular, and join on continuously to zero boundary values.

By means of the customary analysis with Green's theorem, we find as a necessary condition on $v(x, t)$ that it satisfy the equation

$$\begin{aligned}(8)\qquad v(x_1,t_1) &= \frac{1}{2\sqrt{\pi}}\iint_{t_0\Omega t_1} g(x,t|x_1,t_1)\left\{A(x,t,t)v(x,t)\right. \\ &\qquad \left. - \int_{T_x}^{t} A_1(x,t,\tau)v(x,\tau)\,d\tau\right\}dx\,dt \\ &\quad + \frac{1}{2\sqrt{\pi}}\iint_{t_0\Omega t_1} g(x,t|x_1,t_1)\left\{A(x,t,t)u'(x,t)\right. \\ &\qquad \left. - \int_{T_x}^{t} A_1(x,t,\tau)u'(x,\tau)\,d\tau\right\}dx\,dt,\end{aligned}$$

whence, if we add $u'(x_1, t_1)$ to both members, it follows that we must have

$$\begin{aligned}(9)\qquad u(x_1,t_1) &= u'(x_1,t_1) + \frac{1}{2\sqrt{\pi}}\iint_{t_0\Omega t_1} g(x,t|x_1,t_1)\left\{A(x,t,t)u(x,t)\right. \\ &\qquad \left. - \int_{T_x}^{t} A_1(x,t,\tau)u(x,\tau)\,d\tau\right\}dx\,dt.\end{aligned}$$

This, by a permissible change in the order of integration, since $g(x, t|x_1, t_1)$ is a continuous function divided by $\sqrt{t_1 - t}$, may be written in the form

$$(10)\qquad u(x_1,t_1) = u'(x_1,t_1) + \iint_{t_0\Omega t_1} K(x_1,t_1|x,t)\,u(x,t)\,dx\,dt,$$

where $K(x_1, t_1|x, t)$ is given by (4). It is necessary then that $u(x_1, t_1)$ be a solution of (10) within the region.

On the other hand if $u(x_1, t_1)$ is continuous and a solution of (10), it will be regular and a solution of (1), and will join on continuously to the boundary values (α). In fact

$$\frac{\partial u}{\partial x_1} = \frac{\partial u'}{\partial x_1} + \frac{1}{2\sqrt{\pi}} \int\int_{t_0 \Omega t_1} \frac{\partial g(x, t | x_1, t_1)}{\partial x_1} \left\{ A(x, t, t) u(x, t) - \int_{T_x}^{t} A_1(x, t, \tau) u(x, \tau) d\tau \right\} dx\, dt,$$

and is therefore finite and continuous. Hence if we put

$$(11) \qquad \phi(x, t) = A(x, t, t) u(x, t) - \int_{T_x}^{t} A_1(x, t, \tau) u(x, \tau) d\tau$$

we see that

$$\frac{\partial \phi(x, t)}{\partial x} = \frac{\partial}{\partial x}[A(x, t, t) u(x, t)] - \int_{T_x}^{t} \frac{\partial}{\partial x}[A_1(x, t, \tau) u(x, \tau)] d\tau + A_1(x, t, T_x) u(x, T_x) \frac{dT_x}{dx}$$

represents a function of x, t which is continuous except for discontinuities regularly distributed. On some of these lines of discontinuity $\partial\phi/\partial x$ will become infinite. But referring to the theorem of Professor Hurwitz already quoted, we see that the function $\bar{u}(x, t)$ determined by giving $\phi(x, t)$ the value from (11) will be regular and a solution of equation (7), which takes on zero boundary values. Moreover, as we see from equation (9) this $\bar{u}(x, t)$ is precisely $u(x, t) - u'(x, t)$ where $u(x, t)$ is the supposed continuous solution of (10), and $u'(x, t)$ is the $u'(x, t)$ already specified. Hence $u(x, t)$ will satisfy (6), and therefore (1), will be regular, and will take on the given boundary values (α). Our assertion is therefore proved.

It may be noticed here that the discontinuities in the derivatives of highest order in the integro-differential equation (1) appear as projected through the region from points on the boundary, on account of the dependence of the equation on the chosen region, a fact which has no analog in the theory of the parabolic differential equation.

4. The integral equation (10) and its solution in closed form. It merely remains to show that (10) has a unique continuous solution, which can be written in the form (2).

We can show directly that equation (10) has one and only one continuous solution. In fact, the solution may be written in the form of an absolutely and uniformly convergent series of continuous functions*

* Evans, *Sull'equazione integro-differenziale di tipo parabolico*, Rendiconti della R. Accademia dei Lincei, vol. 21 (1912), p. 28. The proof applies with obvious modifications to this case.

$$u(x_1, t_1) = \sum_{n=1}^{\infty} u^{(n)}(x_1, t_1), \tag{12}$$

where

$$u^{(n)}(x_1, t_1) = \int\int_{t_0\Omega_{t_1}} K(x_1, t_1|x, t)\, u^{(n-1)}(x, t)\, dx\, dt, \qquad n > 1,$$

$$u^{(1)}(x_1, t_1) = u'(x_1, t_1).$$

In order to reduce the solution to the closed form (3), we go through a process which incidentally gives us another proof of its existence. We must determine the function $k(x_1, t_1|x, t)$ which satisfies Volterra's relation (equation (5)).

To this purpose let us consider the series formed in the usual way

$$-\sum_{1}^{\infty} K_i(x_1, t_1|x, t) \tag{13}$$

in which

$$K_1(x_1, t_1|x, t) = K(x_1, t_1|x, t)$$

$$K_i(x_1, t_1|x, t) = \int\int_{t\Omega_{t_1}} K(x_1, t_1|\xi, \tau)\, K_{i-1}(\xi, \tau|x, t)\, d\xi\, d\tau.$$

Of this series every term but the first remains finite. In fact for the second term, since from the explicit expression for $K(x_1, t_1|x, t)$

$$|K(x_1, t_1|\xi, \tau)| \leqq \sqrt{\frac{M}{t_1 - \tau}},$$

in which M is some constant, we have

$$\int\int_{t\Omega_{t_1}} |K(x_1, t_1|\xi, \tau)\, K(\xi, \tau|x, t)|\, dt \leqq M^2 A \int_t^{t_1} \frac{d\tau}{\sqrt{t_1 - \tau}\sqrt{\tau - t}} \tag{14}$$

$$\leqq \pi M^2 A,$$

where A is some constant. Hence that term remains finite. The $(n+2)$th term is in absolute value not greater than the expression

$$\frac{C\{D(t_1 - t)\}^n}{n!}, \qquad n \geqq 0,$$

where C and D are certain constants,—provided of course that we are dealing with integrable functions. If we neglect for the moment this proviso, we see that the series, starting with the second term, is absolutely and uniformly convergent.

The first term is integrable over the given x, t region for any given values

of x_1, t_1, and is continuous in the neighborhood of any point $(x_1, t_1|x, t)$ provided that we do not have both $x_1 = x$, and $t_1 = t$. It is understood of course that $t_1 \geqq t$. Let us now consider the remaining terms in regard to continuity, and incidentally therefore in regard to integrability.

The second term, as we have shown, remains finite, and is continuous except when we happen to have $x_1 = x$, $t_1 = t$. Let us then examine the value of this term in the four dimensional neighborhood of a point

$$(X, T|X, T),$$

where (X, T) is within Ω, and show that by taking our neighborhood small enough, we can make the value of this term as small as we please ($\leqq \eta$) provided that we keep $t_1 \geqq t$; in other words, that the second term is continuous at any point $(X, T|X, T)$, if it is given there the value zero.

By referring to the explicit expression for $K(x_1, t_1|x, t)$, we see that the only term in $K_2(x_1, t_1|x, t)$ that requires examination is

$$\frac{1}{4\pi}\int_{t}^{t_1}\int_{\Omega} \frac{A(x, \tau, \tau)\, A(x, t, t)}{\sqrt{t - \tau}\sqrt{\tau - t}}\, e^{-(x_1-\xi)^2/4(t_1-\tau)-(\xi-x)^2/4(\tau-t)}\, d\xi\, d\tau .$$

Let us divide the region of integration for ξ into three parts, c_1 to $X - \Delta/2$, $X - \Delta/2$ to $X + \Delta/2$, and $X + \Delta/2$ to c_2, where c_1 and c_2 are respectively the minimum and maximum values of x in Ω. If we write $|A(x, t, t)| < N$, and make use of the inequality (14), we see that the above expression is in absolute value less than the expression

$$\frac{N^2\Delta}{4} + \frac{N^2}{4\pi}\int_{c_1}^{X-\Delta/2} d\xi \int_{t}^{t_1} \frac{e^{-(x_1-\xi)^2/4(t_1-\tau)-(\xi-x)^2/4(\tau-t)}}{\sqrt{t_1 - \tau}\sqrt{\tau - t}}\, d\tau$$

$$+ \frac{N^2}{4\pi}\int_{X+\Delta/2}^{c_2} d\xi \int_{t}^{t_1} \frac{e^{-(x_1-\xi)^2/4(t_1-\tau)-(\xi-x)^2/4(\tau-t)}}{\sqrt{t_1 - \tau}\sqrt{\tau - t}}\, d\tau .$$

Let us now choose $|x_1 - X|$ and $|x - X| \leqq \Delta/4$, and Δ small enough so that the first term of the expression is less than $\eta/3$. In the two integrals, the values of $|x_1 - \xi|$ and $|\xi - x|$ will always be at least as great as $\Delta/4$, and the integrands therefore less in absolute value than a certain constant, independent of the choice of x and x_1 within the given interval. Hence if we take $|t_1 - T|$ and $|t - T| < \epsilon$, keeping $t_1 \geqq t$, we may choose ϵ small enough to make each integral separately in absolute value less than $\eta/3$, and thus verify the desired inequality.

Accordingly the second term in (13) is continuous within the region. Hence, as is immediately verifiable, all later terms are continuous in the same region.

If now we define $k(x_1, t_1|x, t)$ as the series (13), it will obviously satisfy

the symmetrical relations (5). The continuous function $u(x, t)$ defined by (2) is now seen to be the same as the $u(x, t)$ determined by (10); for from the symmetry of the relations (5) and of the equations (2) and (10), it follows, in the usual way, that the $u(x, t)$ defined by (2) is the unique continuous solution of (10). Thus the closed form of the solution is established.

2. The Generalized Differential and Integro-Differential Equations of Parabolic Type

5. Introduction. The complication of the methods of §§ 1–4, which are the classical methods, is due to the fact that we must insure the existence of $\partial^2/\partial x^2$ and $\partial/\partial t$ for functions which we wish to consider as possible solutions of any of the various parabolic equations. The possibility of avoiding these complications is suggested by trying to consider the complex of derivatives of highest order as a single differential operator

$$\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t},$$

rather than as the combination of two.

Such a definition may be reached in several ways: we may for instance in forming this expression require that the increments of x and t continue to satisfy a certain relation as they approach zero.* It is more convenient for us, however, because closer to a possible physical interpretation and less laborious in its analysis, to define the operator by substituting for the parabolic differential equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x, t) \tag{1}$$

the generalized equation

$$\int_s \left[\frac{\partial u}{\partial x} dt + u dx \right] = \int\int_\sigma f(x, t)\, dx\, dt \tag{1'}$$

in which s is an arbitrary closed curve of a kind later to be described, and σ is the enclosed region.† Similarly for the integro-differential equation

* We can generalize Green's theorem so that it applies to this operator. The corresponding operator has been defined by H. Petrini in the elliptic case for Poisson's equation, *Les dérivées premiers et secondes du potentiel*, Acta Mathematica, vol. 31 (1908), p. 127–332, see introduction and page 181.

† Compare Professor Bôcher's treatment of the equation $\Delta^2 u = 0$ by means of the relation

$$\int_s \partial u / \partial n = 0.$$

M. Bôcher, *On harmonic functions in two dimensions*, Proceedings of the American Academy of Science, vol. 41 (1905–06).

We may regard the passage from equation (1) to equation (2) as equivalent to defining

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) = \lim_{\sigma = 0} \frac{1}{\sigma} \int_s \left[\frac{\partial}{\partial x} dt + (\ )\, dx \right].$$

The corresponding definition is used by Ignatowsky, *Die Vektoranalysis*, Leipzig (1909–10), in his treatment of Δ^2, but his treatment is not exact at all points.

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \int_{T_x}^{t} A(x, t, \tau)\, u(x, \tau)\, d\tau$$

we can substitute the generalized equation

$$\int_s \left[\frac{\partial u}{\partial x} dt + u dx \right] = \int\int_\sigma d\sigma \int_{T_x}^{t} A(x, t, \tau)\, u(x, \tau)\, d\tau .$$

By establishing the corresponding generalization of Green's theorem we can give a treatment of the boundary value problems for these equations. It is not necessary that any derivative of $f(x, t)$ should exist, or that $f(x, t)$ should satisfy any restriction of the type of Levi's condition,*—a result that applies also to the corresponding generalization of the non-homogeneous elliptic equation. Likewise in the treatment of the integro-differential equation, it is unnecessary that any derivative of $A(x, t, \tau)$ should exist, or that $A(x, t, \tau)$ should satisfy Levi's condition.

6. Regular functions, standard curves, and Green's theorem. We say that a function is regular with reference to a generalized parabolic equation and a given region if with its first derivative with respect to x, it is continuous within and on the boundary of the region. The generalized parabolic equations are defined with reference to the integration of regular functions along certain classes of curves within the given region. These curves may be called standard curves and defined as follows.

A standard curve is a closed curve which does not cut itself at any point, composed of a finite number of branches. For each branch, the coördinates of a point are given by two functions $\phi(q)$ and $\psi(q)$, throughout a finite interval for q; $\phi(q)$ and $\psi(q)$ are assumed to be continuous throughout the closed interval, and therefore finite, with their derivatives of the first order. It is assumed that $\phi'(q)$ and $\psi'(q)$ vanish only at a finite number of points, and do not both vanish at the same point. Hence no line parallel to the x-axis or t-axis can cut a standard curve in more than a finite number of points, unless it includes itself a branch of the curve.

We are now in a position to state Green's theorem:

If u and v are regular, and f and g are continuous, throughout a given simply connected region, and if for any standard curve s contained wholly within the region

$$\int_s \left[\frac{\partial u}{\partial x} dt + u dx \right] = \int\int_\sigma f d\sigma$$

(2)

$$\int_s \left[-\frac{\partial v}{\partial x} dt + v dx \right] = \int\int_\sigma g d\sigma$$

then

$$\int_s \left[\left(v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} \right) dt + uv dx \right] = \int\int_\sigma (vf + ug)\, dx\, dt . \tag{3}$$

* Levi, loc. cit., § 8, Art. 23.

7. Proof of Green's theorem. Let us form the functions

$$U = \int_0^{t'} dt' \int_0^{x'} u(x+x', t+t')\, dx', \qquad V = \int_0^{t''} dt'' \int_0^{x''} v(x+x'', t+t'')\, dx'',$$

$$F = \int_0^{t'} dt' \int_0^{x'} f(x+x', t+t')\, dx', \qquad G = \int_0^{t''} dt'' \int_0^{x''} g(x+x'', t+t'')\, dx''$$

keeping $|t'| < a$, $|x'| < a$, $|t''| < a$, $|x''| < a$, where a is a positive constant less than the smallest distance of any point of s to the boundary of the given region.

If we displace the standard curve s parallel to itself by an amount whose components are x', t', it will still remain within the region, and equation (1) will give us, by the corresponding change of variable,

$$\int_s \left[\frac{\partial u(x+x', t+t')}{\partial x}\, dt + u(x+x', t+t')\, dx \right] = \int\int_\sigma f(x+x', t+t')\, dx\, dt.$$

If we integrate this equation from 0 to x' in regard to x' and from 0 to t' in regard to t' we shall have

$$(4) \qquad \int_s \left[\frac{\partial U(x, x'; t, t')}{\partial x}\, dt + U(x, x'; t, t')\, dx \right] = \int\int_\sigma F(x, x'; t', t')\, dx\, dt,$$

since all the functions involved are continuous. But by a change of variable $\xi = x + x'$, $\eta = t + t'$, the expression for U becomes

$$U = \int^{t+t'} d\eta \int_x^{x+x'} u(\xi, \eta)\, d\xi,$$

so that we have the formulæ

$$\frac{\partial U}{\partial x} = \int_t^{t+t'} \{u(x+x', \eta) - u(x, \eta)\} d\eta,$$

$$\frac{\partial U}{\partial t} = \int_x^{x+x'} \{u(\xi, t+t') - u(\xi, t)\} d\xi,$$

$$\frac{\partial^2 U}{\partial x^2} = \int_t^{t+t'} \left\{ \frac{\partial u(x+x', \eta)}{\partial x} - \frac{\partial u(x, \eta)}{\partial x} \right\} d\eta,$$

all functions which are finite and continuous in σ and on s. Hence

$$\int\int_\sigma \left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} \right) d\sigma = \int_s \left[\frac{\partial U}{\partial x}\, dt + U\, dx \right].$$

But F is continuous in σ, and therefore from (4) we have

$$\frac{\partial U(x, x'; t, t')}{\partial t} - \frac{\partial^2 U(x, x'; t, t')}{\partial x^2} = F(x, x'; t, t').$$

Similarly, we find the equation

$$\frac{\partial V(x, x''; t, t'')}{\partial t} + \frac{\partial^2 V(x, x''; t, t'')}{\partial x^2} = G(x, x''; t, t'').$$

From these two equations it follows by means of Green's theorem, as ordinarily stated, that

$$\begin{aligned}\int_s \Big[\Big\{ & V(x, x''; t, t'') \frac{\partial U(x, x'; t, t')}{\partial x} \\ & - U(x, x'; t, t') \frac{\partial V(x, x''; t, t'')}{\partial x} \Big\} dt + U(x, x'; t, t') \\ & \times V(x, x''; t, t'') dx \Big] = \int\int_\sigma \Big\{ V(x, x''; t, t'') F(x, x'; t, t') \\ & + U(x, x'; t, t') G(x, x''; t, t'') \Big\} d\sigma. \end{aligned} \tag{5}$$

Referring to the definitions of U and V, F and G, we see that

$$\frac{\partial^2 U}{\partial x' \partial t'} = u(x + x', t + t'), \qquad \frac{\partial^2 V}{\partial x'' \partial t''} = v(x + x'', t + t''),$$

$$\frac{\partial^2 F}{\partial x' \partial t'} = f(x + x', t + t'), \qquad \frac{\partial^2 G}{\partial x'' \partial t''} = g(x + x'', t + t'').$$

Hence if we differentiate (5) in regard to x', t', x'', t'', under the integral sign, as we have a right to do, since the resulting functions are all finite and continuous, we get the formula

$$\begin{aligned}\int_s \Big[\Big\{ & v(x + x'', t + t'') \frac{\partial u(x + x', t + t')}{\partial x} \\ & - u(x + x', t + t') \frac{\partial v(x + x'', t + t'')}{\partial x} \Big\} dt + u(x + x', t + t') \\ & \times v(x + x'', t + t'') dx \Big] = \int\int_\sigma \Big\{ v(x + x'', t + t'') \\ & \times f(x + x', t + t') + u(x + x', t + t') g(x + x'', t + t'') \Big\} d\sigma. \end{aligned}$$

If in this we put $x' = x'' = t' = t'' = 0$ we get equation (3), which we wished to prove.

If the boundary of the region is a standard curve the theorem will still hold if we

take s as the boundary itself. This can be shown by extending the function u across s in the following manner.

Let us isolate each vertex of the curve, and each point of the curve where there is a vertical tangent, unless it is an interior point of a vertical branch of the curve, by a small circle. Consider the portion of the curve between two successive circles. If this portion is vertical, we shall define u and $\partial u/\partial x$ throughout a small neighborhood on the outside of the curve by means of the values which they possess on the vertical line itself for the same value of x.

If the portion is not vertical it will not have a vertical tangent at any interior point. Let us denote by $r(t)$ the values of $\partial u/\partial x$ along this portion of the curve, and define u throughout a small neighborhood on the outside of the curve as that solution of the equation

$$\frac{\partial u}{\partial x} = r(t)$$

which takes on along the curve itself the already known values of u.

If now we surround each circle by a slightly larger circle, and take as a contour s' the curve composed, first, of the portions of s between the successive circles, last constructed, and second of the portions of these circles interior to the region, we shall have for s', if we take the circles small enough, a standard curve, and it will lie wholly within a region for which u and $\partial u/\partial x$ are continuous. Green's theorem will then hold for s'. We get Green's theorem for s if we let the circles approach zero as a limit.

8. The generalized parabolic differential equation. As the region to be considered, we shall understand that enclosed by the lines $t = t_0$ on the left, and $t = t_1$ on the right, and by the curves $x = \xi_2(t)$ above and $x = \xi_1(t)$ below. We shall assume that $\xi_1(t)$, $\xi_2(t)$ and their first derivatives remain single-valued, finite, and continuous, and that $\xi_2(t) > \xi_1(t)$, $t \geqq t_0$. The functions $\xi_1(t)$ and $\xi_2(t)$ are to have only a finite number of maxima and minima in the interval $t_0 \leqq t \leqq t_1$.

Along the open boundary is to be given the continuous chain of values (α) of § 1, under the conditions there stated. The function $f(x, t)$ is to be finite, and continuous within the given region.

Under these conditions there is one and only one regular solution $u(x, t)$ of the generalized parabolic differential equation (1'), *which joins on continuously to the boundary values* (α). *It has the form*

$$(6) \qquad u(x_1, t_1) = u'(x_1, t_1) + \frac{1}{2\sqrt{\pi}} \int_{t_0}^{t_1} \int_{\Omega} g(x, t | x_1, t_1) f(x, t)\, dx\, dt$$

where $u'(x_1, t_1)$ is the regular solution of the equation

$$(7) \qquad \frac{\partial u'(x, t)}{\partial t} = \frac{\partial^2 u'(x, t)}{\partial x^2}.$$

which joins on continuously to the boundary values (α),* *and* $g(x, t|x_1, t_1)$ *is the Green's function for this same equation.*

From Green's theorem it follows that under the given conditions the solution of (1′) must necessarily have the form (6). On the other hand we can show directly that (6) does give us a solution which takes on the boundary values (α). In fact we shall show this if we show that *the function*

$$z(x_1, t_1) = \frac{1}{2\sqrt{\pi}} \int\int_{t_0\Omega t_1} g(x, t|x_1, t_1) f(x, t)\, dx\, dt \tag{8}$$

is regular with respect to (1′), *takes on continuously the value zero on the open boundary, and satisfies* (1′).

Let us write

$$z(x, \tau) = Z(x, t) + w(x, t)$$

and

$$Z(x_1, t_1) = \frac{1}{2\sqrt{\pi}} \int\int_{t_0\Omega t_1} \frac{1}{\sqrt{t_1 - t}} e^{-(x_1 - x)^2/4(t - t_1)} f(x, t)\, dx\, dt.$$

We know that $Z(x_1, t_1)$ is regular with regard to (1′) since with its derivative in regard to x it is continuous within and on the boundary of Ω.† It is also a solution of (1′), as we can show by performing the indicated integration.

9. Formulæ of integration. Let l be a straight line lying wholly within Ω, of which the end points are (a_1, b_1) and (a_2, b_2), with $b_2 > b_1$, and let x' be the x coördinate of the point on that line whose other coördinate is t, i. e.

$$x' = \frac{a_2 - a_1}{b_2 - b_1}(t - b_1) + a_1.$$

If we enclose this line in a small parallelogram, which we allow to vanish, we can easily deduce the following formulæ.

If l is perpendicular to the t-axis, we have

$$\int_{a_1}^{a_2} dx_1 \int\int_{t_0\Omega t_1} h_{0\,1/2}(x, t|x_1, t_1) f(x, t)\, dx\, dt = \lim_{\delta=0} \int\int_{t_0\Omega t_1-\delta} dx\, dt\, f(x, t) \int_{a_1}^{a_2} h_{0\,1/2}(x, t|x_1, t_1)\, dx_1. \tag{I}$$

If l is not perpendicular to the t-axis, we have

* For the existence and uniqueness of $u'(x, t)$ under these conditions see G. E. Levi, loc. cit., §§ 2, 5.

† See E. E. Levi, loc. cit., p. 234. See also § 7.

$$(\mathrm{II}_a)\qquad \int_{a_1{}^l a_2} dx_1 \iint_{t_0\Omega t_1} h_{0\,1/2}(x,t|x_1,t_1) f(x,t)\,dx\,dt = \iint_{t_0\Omega b_2} dx\,dt f(x,t) \int_{x^l a_2} h_{0\,1/2}(x,t|x_1,t_1)\,dx_1,$$

$$(\mathrm{II}_b)\qquad \int_{b_1{}^l b_2} dt_1 \iint_{t_0\Omega t_1} h_{1\,3/2}(x,t|x_1,t_1) f(x,t)\,dx\,dt = \iint_{t_0\Omega b_2} dx\,dt f(x,t) \int_{t^l b_2} h_{1\,3/2}(x,t|x_1,t_1)\,dt_1.$$

In these formulæ, as before, we write

$$h_{\alpha\beta}(x,t|x_1,t_1) = \frac{(x_1-x)^\alpha}{(t_1-t)^\beta}\, e^{(x_1-x)^2/4(t_1-t)},$$

from which it follows that

$$\frac{\partial}{\partial x_1} h_{0\,1/2}(x,t|x_1,t_1) = -\tfrac{1}{2}\, h_{1\,3/2}(x,t|x_1,t_1).$$

10. Application to equation (1′). With reference to these formulæ let us form the expression

$$\int_C \left[\frac{\partial Z}{\partial x_1} dt_1 + Z dx_1\right],$$

where C denotes a closed polygon of a finite number of sides, the boundary of which does not anywhere cut itself. If we denote by ${}_tC$ the portion of the contour composed of points of which the t-coördinates are *greater* than the value t, and by b the maximum t-coördinate for any point of the contour, we shall have the equation

$$\int_C \left[\frac{\partial Z(x_1,t_1)}{\partial x_1} dt_1 + Z(x_1,t_1)\,dx_1\right] = \frac{1}{2\sqrt{\pi}} \iint_{t_0\Omega b} dx\,dt f(x,t) \int_{{}_tC} \left[\frac{\partial h_{0\frac{1}{2}}(x,t|x_1,t_1)}{\partial x_1} dt_1 + h_{0\,1/2}(x,t|x_1,t_1)\,dx_1\right].$$

The curvilinear integral of the second member of the above equation is however, as is well known, equal to

$$2\sqrt{\pi} \quad \text{if the point } (x,t) \text{ is inside } C,$$

$$0 \quad \text{if the point } (x,t) \text{ is outside } C,$$

since $h_{0\frac{1}{2}}$ is itself a solution in x_1, t_1 of the homogeneous partial differential equation (7). The boundary points form an aggregate of zero external

measure, and since the curvilinear integral remains finite, contribute nothing to the value of the second member. If then we divide ${}_{t_0}\Omega_{t_1}$ into two parts, one consisting of the portion of Ω inside C, and the other the portion of Ω outside C, we observe that the value of the double integral over the latter portion is zero, on account of the zero factor in the integrand. Hence we have

$$\int_C \left[\frac{\partial Z(x_1, t_1)}{\partial x_1} dt_1 + Z(x_1, t_1) dx_1 \right] = \int\int_C f(x, t) dx\, dt,$$

the double integral being extended over the interior of C.

Let us now finally consider any standard curve entirely within Ω. In the first place it is rectifiable, and its length is by definition the limit of the perimeter of an inscribed polygon as the vertices of the polygon become infinite in number, in such a way as to make the polygon approach the contour. In order to construct the approximate polygons in a perfectly definite manner, we may imagine the ith branch of the standard curve s to have the coördinates of its points expressed in terms of the parameter q, as we have already described, and take as the vertices of the polygon the n_i *interior* points of division corresponding to equally spaced values of q in the interval (the vertices of the curve therefore are not vertices of the polygon). By taking every n_i large enough, we get in this way a polygon of the kind already described, namely one of a finite number of sides which does not cut itself. If we denote the total approximate polygon by C_n, we shall have

$$\int_{C_n} \left[\frac{\partial Z}{\partial x} dt + Z dx \right] = \int\int_{C_n} f(x, t) dx\, dt.$$

As each n_i becomes infinite the right-hand member approaches the double integral over σ. The left-hand member approaches the integral over s, since the functions Z and $\partial Z/\partial x$ are finite and continuous, and the curve s is rectifiable. In fact the integral around s may be regarded as the limit of a sum, the values of the functions being taken at the vertices of the polygons; and from the continuity of the functions u and $\partial u/\partial x$ in the two-dimensional region it follows that the integral around C_n can also be made to differ by as little as we please from that sum, and from its limit, by taking all the n_i large enough.

The function Z is therefore a solution of equation (1′).

Let $w(x, t)$ be that solution of (7) which has on the open boundary of Ω the same values as $Z(x, t)$, reversed in sign. Since $Z(x, t)$ is finite and continuous within and on the boundary of Ω, $w(x, t)$ will be regular with respect to (7), as the existence theorem for (7) tells us. The function

$$z(x, t) = Z(x, t) + w(x, t)$$

will therefore be regular with respect to (1′), will take on zero values on the boundary of Ω, and will satisfy (1′), for any standard curve lying wholly within the boundary. It follows moreover from Green's theorem that it must be expressible in the form (8). This is what was to be proved.

11. A generalized integro-differential equation. Let us consider now an integro-differential equation of parabolic type corresponding to the region Ω. If we denote by $\int_{t_0 t_1}$ integration along that portion of a line parallel to the axis of t which is inside the region Ω and between the lines $t = t_0$ and $t = t_1$, the integral equation that we desire to consider may be written

$$\int_s \left[\frac{\partial u}{\partial x}dt + u dx\right] = \iint_\sigma dx\,dt \int_{t_0 t} A(x, t, \tau)\, u(x, \tau)\, d\tau, \tag{9}$$

where by s is meant any closed contour of the kind that we have already specified.

If $u(x, t)$ is continuous, the quantity

$$\int_{t_0 t} A(x, t, \tau)\, u(x, \tau)\, d\tau$$

will be a continuous function of x and t.* Hence by the theorem of § 8, in order that $u(x, t)$ be the regular solution of (9), which takes on the boundary conditions (α) it is necessary and sufficient that it be a continuous solution of the equation

$$\begin{aligned} u(x_1, t_1) &= u'(x_1, t_1) \\ &\quad + \frac{1}{2\sqrt{\pi}} \iint_{t_0 \Omega_{t_1}} dx\,dt\, g(x, t | x_1, t_1) \int_{t_0 t} A(x, t, \tau)\, u(x, \tau)\, d\tau. \end{aligned} \tag{10}$$

This equation, however, is merely an integral equation similar to that already discussed in part 1. For by rewriting this equation in the form

$$\begin{aligned} u(x_1, t_1) &= u'(x_1, t_1) \\ &\quad + \frac{1}{2\sqrt{\pi}} \int_{c_1}^{c_2} dx \int_{t_0 t_1} dt\, g(x, t | x_1, t_1) \int_{t_0 t} A(x, t, \tau)\, u(x, \tau)\, d\tau, \end{aligned}$$

* According to our conditions the boundaries $x = \xi_1(t)$ and $x = \xi_2(t)$ cannot have portions consisting of horizontal straight lines. If it is desired to admit these, we must also generalize the theorem of § 4 to apply to functions $f(x, t)$ which are finite and discontinuous in a regular manner, — a generalization which is immediate.

and interchanging the order of integration with respect to t and τ we have the equation

$$u(x_1, t_1) = u'(x_1, t_1) + \frac{1}{2\sqrt{\pi}} \int_{c_1}^{c_2} dx \int_{t_0 t_1} d\tau\, u(x, \tau) \int_{\tau t_1} g(x, t | x_1, t_1)\, A(x, t, \tau)\, dt,$$

which is again the same as

$$u(x_1, t_1) = u'(x_1, t_1) + \frac{1}{2\sqrt{\pi}} \iint_{t_0 \Omega t_1} dx\, dt\, u(x, t) \int_{t t_1} g(x, \tau | x_1, t_1)\, A(x, \tau, t)\, d\tau. \tag{11}$$

But this equation, according to the method of part 1, has a unique continuous solution of the kind there specified. We have then the following

THEOREM. *There is one and only one regular solution $u(x, t)$ of the equation* (9) *which joins on continuously to the boundary values* (α). *It may be reduced to the form*

$$u(x_1, t_1) = u'(x_1, t_1) - \iint_{t_0 \Omega t_1} k(x_1, t_1 | x, t)\, u'(x, t)\, dx\, dt,$$

where $k(x_1, t_1 | x, t)$ is the function associated to

$$K(x_1, t_1 | x, t) = \frac{1}{2\sqrt{\pi}} \int_{t t_1} g(x, \tau | x_1, t_1)\, A(x, \tau, t)\, d\tau,$$

by means of Volterra's relation.

3. The Integro-Differential Equation of Hyperbolic Type

12. In this last section we consider very briefly the hyperbolic integro-differential equation of the following simple type,*

$$\frac{\partial^2 u(x, t)}{\partial x^2} - \frac{\partial^2 u(x, t)}{\partial t^2} = \int_0^t A(x, t, \tau)\, \frac{\partial^2 u(x, \tau)}{\partial \tau^2}\, d\tau, \tag{1}$$

in which $A(x, t, \tau)$ is a continuous function, with continuous first derivatives, in a region

$$0 \leqq \tau \leqq t \leqq r$$

$$a - r + t \leqq x \leqq a + r - t.$$

* This equation has been treated by the method of partial solutions by V. Volterra, *Vibrazioni elastiche nel caso della eredità*, Rendiconti della R. Accademia dei Lincei, vol. 21 (1912), p. 1. See also V. Volterra, *Lectures delivered at the celebration of the 20th anniversary of the foundation of Clark University* (1911), p. 80.

The region of variation for the point (x, t) is thus a triangle whose vertex is at the point (a, r) and whose base is on the x-axis bounded by the points $(a - r, 0)$ and $(a + r, 0)$. The solution of this equation is to be regular,* and is to satisfy the boundary conditions

$$u(x, 0) = f(x),$$

$$(\beta) \qquad \left[\frac{\partial u(x, t)}{\partial t}\right]_{t=0} = f_1(x),$$

in which $f(x)$, $f_1(x)$ and their first derivatives are continuous functions of x in the range of values $a - r \leqq x \leqq a + r$.

As long as we base our treatment on the characteristic function of Riemann, which is itself regular, there is no need of generalizing the equation in order to get rid of the second derivatives. We therefore limit ourselves in this section to a treatment that corresponds to § 1, and content ourselves with a bare mention of the resulting theorem.

THEOREM. *Under the given conditions for $A(x, t, \tau)$ there is one and only one regular solution of (1), $u(x, t)$, which joins on continuously to the boundary values (β). It may be reduced to the form*

$$(2) \qquad u(x, t) = u'(x, t) - \int_{x-t}^{x+t} d\xi \int_0^{\tau_x} l(x, t|\xi, \tau)\, u'(\xi, \tau)\, d\tau,$$

where

$$\tau_x = t - x + \xi \quad \text{for} \quad \xi \leqq x$$

$$\tau_x = t + x - \xi \quad \text{for} \quad \xi \geqq x,$$

where $u'(x, t)$ is the regular solution of the equation

$$\frac{\partial^2 u'(x, \tau)}{\partial x^2} - \frac{\partial^2 u'(x, t)}{\partial t^2} - A(x, t, \tau)\frac{\partial u'(x, t)}{\partial t}$$

$$(3) \qquad + \left[\frac{\partial A(x, t, \tau)}{\partial \tau}\right]_{\tau=t} u'(x, \tau)$$

$$= -A(x, t, 0) f_1(x) + \left[\frac{\partial A(x, t, \tau)}{\partial \tau}\right]_{t=0} f(x)$$

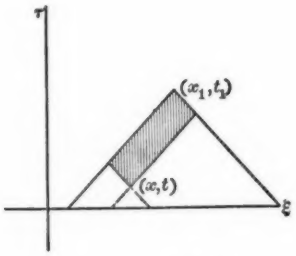
that takes on the boundary conditions (β), and where $l(x, t|\xi, \tau)$ is the function associated to

$$(4) \qquad L(x_1, t_1|x, t) = \frac{1}{2}\int_t^{t_x} G(x_1, t_1|x, \tau)\frac{\partial^2 A(x, t, \tau)}{\partial t^2}\, d\tau$$

* That is, is to be continuous, with continuous first derivatives, and integrable second derivatives ($\partial^2/\partial x^2$ and $\partial^2/\partial t^2$) in the given region $0 \leqq t \leqq r$, $a - r + t \leqq x \leqq a + r - t$.

by means of Volterra's relation. By $G(x_1, t_1|x, t)$ *is denoted Riemann's characteristic function for equation* (3).*

In the case of the region that we are considering Volterra's relation may be written as



$$(5)\quad l(x_1, t_1|x, t) + L(x_1, t_1|x, t) = \int\int L(x_1, t_1|\xi, \tau)\, l(\xi, \tau|x, t)\, d\xi\, d\tau,$$

the double integration being extended over the shaded area in the accompanying diagram.

THE RICE INSTITUTE,
June, 1914.

* That is, the solution of the equation adjoint to the first member of (3), which takes on the value 1 for $x = x_1$, $y = y_1$, and satisfies the conditions

$$\sqrt{2}\frac{dG}{ds_1} = \frac{A(x, t, t)}{2} G \text{ along the line } x - t = x_1 - t_1$$

and

$$\sqrt{2}\frac{dG}{ds_2} = \frac{A(x, t, t)}{2} G \text{ along the line } x + t = x_1 + t_1.$$

INVARIANTS IN THE THEORY OF NUMBERS*

BY

L. E. DICKSON

1. Polynomials in the coefficients of a form $f(x_1, \cdots, x_n)$ which have the invariantive property with respect to all linear homogeneous transformations on $x_1, \cdots, x_n$ with integral coefficients taken modulo p, where p is a prime, are called formal invariants modulo p of f if the coefficients of f are independent variables, but are called modular invariants if the coefficients of f are integers taken modulo p. The concept of formal invariants modulo p was introduced by Hurwitz;† but the only known results concerning them relate to the binary quadratic and cubic forms.‡ On the contrary, a simple and effective theory of modular invariants has been given by the writer. A new method of deriving modular invariants from seminvariants is given in § 8. But the main purpose of this paper is to present a simple general method of constructing formal invariants. The method is applicable also to formal seminvariants and, more generally, to the invariants of any linear congruence group (§ 7). Moreover, the new point of view forms an adequate basis for a general theory of formal invariants.

Construction of Formal Invariants

2. The method of construction will first be illustrated by the simple example of the binary quadratic form

$$Q = ax^2 + bxy + cy^2$$

for the case of modulus 2. The only real points (i. e., with integral coördinates) modulo 2 are

$$P_1 = (1, 0), \qquad P_2 = (0, 1), \qquad P_3 = (1, 1).$$

The corresponding values of Q are

$$a, \qquad c, \qquad s = a + b + c.$$

By the interchange of x and y in Q, a and c are interchanged, also P_1 and P_2.

* Presented to the Society at Providence, September 7, 1914.

† Archiv der Mathematik und Physik, ser. 3, vol. 5 (1903), p. 25.

‡ Dickson, *On Invariants and the Theory of Numbers, Madison Colloquium Lectures*, American Mathematical Society, pp. 40–54.

By the transformation $x' = x + y$, $y' = y$, Q is replaced modulo 2 by a form with the same a and b, but with c replaced by s; then c and s are interchanged, as well as P_2 and P_3. Since any binary linear transformation with integral coefficients modulo 2 is generated by the preceding two, it follows that any such transformation gives rise to a permutation of a, c, s amongst themselves, and the same permutation of P_1, P_2, P_3. Hence *any symmetric function of a, c, s is a formal invariant modulo 2 of Q.*

The elementary symmetric functions are, modulo 2,

$$b, \qquad q = a^2 + ac + c^2 + (a + c)b, \qquad k = ac(a + b + c).$$

They form* a fundamental system of rational integral formal invariants modulo 2 of Q.

3. Consider the system of forms Q and

$$l = \eta x + \xi y.$$

The values of l at the points P_1, P_2, P_3 are $\eta, \xi, \eta + \xi$, respectively. The latter undergo the same permutation as the P's when l is transformed linearly (§ 2). Hence, if ϕ is any polynomial in two arguments,

$$\phi(a, \eta), \qquad \phi(c, \xi), \qquad \phi(s, \eta + \xi)$$

are permuted amongst themselves when Q and l are transformed linearly modulo 2. Hence any symmetric function of these three ϕ's is a formal invariant modulo 2 of Q and l.

Taking $\phi(a, \eta)$ to be η, $a\eta$, and $a\eta^2$, in turn, and employing the elementary symmetric functions in each case, we get the following formal invariants of Q and l:

$$\lambda = \xi^2 + \xi\eta + \eta^2, \qquad \pi = \xi\eta(\xi + \eta), \qquad j = (a + b)\xi + (b + c)\eta,$$

$$t = ac\xi\eta + s(a\eta + c\xi)(\xi + \eta), \qquad u = (a + b)\xi^2 + (b + c)\eta^2,$$

$$v = ac\xi^2\eta^2 + s(a\eta^2 + c\xi^2)(\xi^2 + \eta^2),$$

and $k\pi$, $k\pi^2$. From $\phi = a\eta^4$, we get

$$U = (a + b)\xi^4 + (b + c)\eta^4 = \lambda u + j\pi.$$

Also t and v can be expressed in terms of simpler invariants:

$$t = q\lambda + j^2 + bu, \qquad v = q\lambda^2 + u^2 + bU.$$

Whether or not the reduced set $b, q, k, \lambda, \pi, j, u$ form a fundamental system of formal invariants modulo 2 of Q and l has not been investigated.

But if we pass to modular invariants by regarding a, b, c, ξ, η to be integers

* *Madison Colloquium Lectures*, p. 42.

taken modulo 2, we have $k \equiv bq$, $\pi \equiv 0$, $u \equiv j$, and the reduced set b, q, λ, j form* a fundamental system of modular invariants of Q and l.

4. Passing to the general case, let

$$f_1(x_1, \cdots, x_n), \quad \cdots, \quad f_t(x_1, \cdots, x_n)$$

be forms of total degrees $d_1, \cdots, d_t$ in the independent variables $x_1, \cdots, x_n$. Let the modulus be p and let g_i be the greatest common divisor of $p-1$ and d_i. Set $q_i = (p-1)/g_i$. Then for any integer ρ not divisible by p,

$$\rho^{d_i q_i} = (\rho^{p-1})^{d_i/g_i} \equiv 1 \qquad (\text{mod } p),$$

by Fermat's theorem. Hence, since f_i is of degree d_i,

$$[f_i(\rho x_1, \cdots, \rho x_n)]^{q_i} \equiv [f_i(x_1, \cdots, x_n)]^{q_i} \qquad (\text{mod } p).$$

Using homogeneous coördinates, let $(x_1, \cdots, x_n)$ be the same point as $(\rho x_1, \cdots, \rho x_n)$, when ρ is any integer not divisible by p. The preceding formula shows that $f_i^{q_i}$ has a definite value at each real point (i. e., one with integral coördinates taken modulo p). The values at the various real points are merely permuted amongst themselves by any linear homogeneous transformation on $x_1, \cdots, x_n$ with integral coefficients taken modulo p. In fact, the real points are permuted by such a transformation. We thus have the

THEOREM. *We obtain a formal invariant modulo p of the system of forms $f_i(x_1, \cdots, x_n)$, $i = 1, \cdots, t$, if we take any symmetric function with integral coefficients of the quantities*

$$\phi(f_1^{q_1}, \cdots, f_t^{q_t})$$

given by substituting in turn for the x's the coördinates of the

$$P = 1 + p + p^2 + \cdots + p^{n-1}$$

real points. Here ϕ is any polynomial in its t arguments with integral coefficients, and q_i is the quotient of $p-1$ by the greatest common divisor of $p-1$ and the degree of f_i.

Each q_i is unity if $p = 2$ (cf. §§ 2, 3); also if $p = 3$ and each f_i is of even degree. For example, if $t = 1$ and

$$f = f_1 = ax^2 + 2bxy + cy^2,$$

and $p = 3$, the values of f at the $P = 4$ real points $(1, 0)$, $(0, 1)$, $(1, 1)$, $(-1, 1)$ are respectively

$$a, \qquad c, \qquad a - b + c, \qquad a + b + c.$$

* *Madison Colloquium Lectures*, p. 57.

The elementary symmetric functions reduce modulo 2 to zero and

$$ac - b^2, \qquad (a+c)(a+b-c)(a-b-c),$$
$$ac(a+b+c)(a-b+c),$$

respectively. The first is the discriminant D of f. The last two are $\gamma_2 = \Gamma$ and $a\gamma_0 = J$ in the notations of the *Madison Colloquium Lectures*, pages 43–45.

A further evident invariant is the product of all non-proportional linear functions of a, b, c with integral coefficients taken modulo 3. Hence its quotient

$$B = b(b+a)(b-a)(c-a)(c-b)(c+b)$$

by ΓJ is a formal invariant. In the place just cited it was proved that D, Γ, J, B form a fundamental system of formal invariants modulo 3 of f.

If $p = 5$, the sum of the products by twos of the squares of the values of f at the six real points is

$$3(ac - b^2)^2 \pmod 5.$$

Hence as before we are led to the algebraic discriminant, as well as to invariants peculiar to the number theory case.

5. The present method is particularly useful for forms in three or more variables, since the construction of formal or modular invariants by earlier methods is excessively laborious and dependent largely upon special devices.

As an illustration, consider

$$F(x) = a_1 x_2 x_3 + a_2 x_1 x_3 + a_3 x_1 x_2 + b_1 x_1^2 + b_2 x_2^2 + b_3 x_3^2$$

for the modulus 2. Its values v_j for the seven real points are

$$b_1, \quad b_2, \quad b_3, \quad a_1+b_2+b_3, \quad a_2+b_1+b_3, \quad a_3+b_1+b_2,$$
$$\sum_{i=1}^{3}(a_i+b_i).$$

Since their sum is zero modulo 2, we readily get

$$\sum v_1 v_2 = \sum a_1 b_1 + \sum a_1^2 + a_1 a_2 + a_1 a_3 + a_2 a_3 = \alpha,$$
$$\sum v_1^2 v_2 = \sum v_1 v_2 v_3 = \sum a_1 b_1^2 + \sum a_1^2 b_1 + \sum a_1 a_2^2,$$
$$\sum v_1 v_2 v_3 v_4 = \alpha \sum b_1^2 + \sum a_1 b_1 (a_1+b_1)(b_2+b_3) + \sum a_1 a_2 b_1 b_2$$
$$+ (a_1 a_2 a_3 + a_1 a_2 b_3 + a_1 a_3 b_2 + a_2 a_3 b_1) \sum a_1$$
$$+ \sum b_1^4 + \sum b_1^2 b_2^2 + b_1 b_2 b_3 \sum b_1.$$

The second is congruent to the sum of the formal invariants

$$\Delta = a_1 a_2 a_3 + \sum a_1^2 b_1, \qquad \Delta_1 = a_1 a_2 a_3 + \sum a_1 b_1^2 + \sum a_1 a_2^2,$$

which, with a modular invariant J, are given on pages 69 and 74 of the *Madison Colloquium Lectures*. Note that Δ is the discriminant of F.

If we take each a_i and b_i to be an integer modulo 2, we find that our third invariant reduces to the modular invariant $J + \alpha + 1$. Set $A = \alpha + \Delta + 1$. It is proved on page 76 of the same book that Δ, A, and J form a fundamental system of modular invariants of F.

As shown indirectly in § 6, we cannot construct a fundamental system of modular invariants of F by use of the real points only, but succeed if we use also points whose coördinates are Galois imaginaries.

Criteria for the Equivalence of Modular Forms

6. An important application of modular invariants is that to the question of the equivalence of two modular forms under linear transformation with integral coefficients taken modulo p. We can treat this question by means of the idea underlying the foregoing construction of invariants, without actually constructing them. For example, consider the ternary quadratic form modulo 2. As well known, such a form is equivalent to one and but one of the forms

$$x_1 x_2 + x_3^2\,[4], \qquad x_1 x_2 + x_1^2 + x_2^2\,[6], \qquad x_1 x_2\,[2], \qquad x_1^2\,[4], \qquad 0\,[0].$$

After each form we have given the number of real points for which it is congruent to unity modulo 2. Hence no two of these forms are equivalent, with a possible exception in the case of the first and fourth. To show by the same method that the latter are not equivalent, we employ the points each of whose coördinates are 0, 1, j or $j+1$, where j is the Galois imaginary for which

$$j^2 + j + 1 \equiv 0 \pmod 2. \tag{1}$$

Now x_1^2 vanishes for just five such points, viz., $(0, 0, 1)$ and $(0, 1, a)$, where $a = 0, 1, j$ or $j + 1$. But $x_1 x_2 + x_3^2$ vanishes for just nine such points, viz., $(0, b, 0)$, $(b, 0, 0)$, $(1, b, b^2)$, where $b = 1$, j, or $j + 1$. Hence the two forms are not equivalent.

The second of our five forms vanishes for just nine such points, viz., $(0, 0, 1)$, $(1, j, a)$, $(1, j + 1, a)$. Finally, $x_1 x_2$ vanishes only for nine such points. Hence the five forms vanish for respectively 6, 8, 4, 2, 14 imaginary points whose coördinates depend upon j. Thus *the classes of conics modulo 2 are completely characterized by the number of their imaginary points whose coördinates are functions of a root of* (1).

We readily find the characteristic invariant I_k for the kth class, i. e., an invariant with the value unity for any form in the kth class and the value zero for the remaining forms. For the class represented by our second form,

I_2 is the sum of the products six at a time of the values of F for the seven real points. For the class represented by $x_1 x_2$, I_3 is the sum of the products by twos of those values. We get $I_1 + I_4$ by using the products by fours; to get I_1 itself, we employ imaginary points as above. Finally,

$$1 + I_1 + I_2 + I_3 + I_4 = I_5.$$

Construction of Formal Seminvariants

7. The preceding method is readily extended to the general case of the formal invariants of any system of forms under any group of linear transformations modulo p.

By way of illustration, consider the form Q of § 2 and the group composed of the identity and

$$T: \qquad x' \equiv x + y, \qquad y' \equiv y \pmod 2.$$

It is therefore the question of the formal seminvariants of Q. Since T leaves unaltered the point P_1 of § 2 and interchanges the points P_2 and P_3, a and any symmetric function of c and s are formal seminvariants. The elementary symmetric functions are $a + b$ and cs. These with a form in fact a fundamental system of formal seminvariants modulo 2 of Q (*Madison Colloquium*, page 42).

Similarly, the system of forms Q and l (§ 3) have the formal seminvariants modulo 2

$$a, \qquad \eta, \qquad \phi(c, \xi) + \phi(s, \eta + \xi), \qquad \phi(c, \xi) \cdot \phi(s, \eta + \xi),$$

where ϕ is any polynomial in its two arguments.

Derivation of Modular Invariants from Seminvariants

8. The method will be illustrated for the binary quadratic form Q (§ 2) and the modulus 2. The coefficients are integers taken modulo 2. To make the treatment self-contained, we shall first derive a fundamental system of modular seminvariants. The transformation

$$(x + ty, y): \qquad x' = x + ty, \qquad y' = y$$

replaces Q by $ax^2 + bxy + c'y^2$, where $c' = c + (a + b)t$. Hence a and b are seminvariants of Q. If $a + b \equiv 1$, we take $t = c$ and have $c' \equiv 0$. If $a + b \equiv 0$, then $c' \equiv c$ and the value of c is given by

$$J = (1 + a + b)c.$$

Since J has the same value for equivalent forms Q, it is a seminvariant. The first column of the following table gives a representative of each class $c_1, \cdots, c_6$ of forms Q.

Representative of the Class	Determined by Seminvariants	Characteristic Seminvariant
$Q_1 = x^2 + xy + y^2$	$a = b = 1,\ J = 1$	$\pi = abJ \equiv abc$
$Q_2 = x^2 + xy$	$a = b = 1,\ J = 0$	$ab(J+1) = \pi + ab$
$Q_3 = y^2$	$a = b = 0,\ J = 1$	$(a+1)(b+1)J = \pi + J$
$Q_4 \equiv 0$	$a = b = 0,\ J = 0$	$(a+1)(b+1)(J+1) = I$
$Q_5 = x^2$	$a = 1,\ b = 0$	$a(b+1)$
$Q_6 = xy$	$a = 0,\ b = 1$	$(a+1)b$

By definition, the characteristic seminvariant of a class c_i is one having the value unity for each form of the class c_i and the value zero for each form in the remaining classes. In the present case, the characteristic seminvariant of class c_i was found by inspection from the values of the seminvariants in the second column which specify that class,—it is a product in which any factor is one of those seminvariants or the sum of it and unity, according as the seminvariant is 1 or 0 for the class.

To derive the invariants of Q from its characteristic seminvariants, we separate the forms Q into classes C_i of equivalent forms under the group G of all binary linear transformations modulo 2. Now G is generated by $(x + y, y)$ and (y, x). Since classes $c_1, \cdots, c_4$ are each composed of a single form, and since the forms Q_1 and Q_4 are unaltered by (y, x), while Q_2 is changed into a form of c_6, and Q_3 into Q_5, we see that the classes under G are

$$C_1 = c_1, \qquad C_2 = c_2 + c_6, \qquad C_3 = c_3 + c_5, \qquad C_4 = c_4.$$

Hence, for $i = 1$ or 4, the characteristic invariant I_i for class C_i is the characteristic seminvariant for c_i. But I_2 is the sum of the characteristic seminvariants for c_2 and c_6, and I_3 the sum of those for c_3 and c_5. Thus the characteristic invariants are

$$I_1 = \pi, \qquad I_2 = \pi + b, \qquad I_3 = \pi + J + ab + a = I + b + 1, \qquad I_4 = I.$$

In § 2 it was shown that b, q, k form a fundamental system of formal invariants modulo 2 of Q. Taking the coefficients to be integers modulo 2, we obtain the modular invariants b, $q \equiv \delta$, $k \equiv b\delta$, where

$$b = I_1 + I_2, \qquad \delta = ac + (b+1)(a+c) = I_1 + I_3.$$

Conversely, from b and δ we get

$$\pi = b\delta, \qquad I = (b+1)(\delta+1).$$

This new method of deriving all modular invariants from the seminvariants is more direct and simpler than the method employed in the *Madison Colloquium*, pages 28–32.

ADDENDA AND ERRATA, VOLUMES, 11, 14

VOLUME 11

Page 25. ERNEST B. LYTLE.* *Proper multiple integrals over iterable fields.*

1. Consider the fundamental relation

$$(A) \qquad \underline{\int}_{\mathfrak{A}} f \leqq \underline{\int}\,\underline{\int}_{\mathfrak{G}} f \leqq \overline{\int}_{\mathfrak{B}}\overline{\int}_{\mathfrak{G}} f \leqq \overline{\int}_{\mathfrak{A}} f$$

where f is any limited function defined over the limited field $\mathfrak{A}$, and where the integrals are Pierpont integrals and hence are applicable to fields which are not metric (defined in these Transactions, vol. 11 (1910), p. 26). The problem under consideration here is to find the most general conditions upon the field $\mathfrak{A}$ under which relation (A) is true.

Pierpont has shown† that relation (A) is true when the field $\mathfrak{A}$ is metric. In a former paper‡ the author found a more general class of fields called iterable for which relation (A) holds true. This class of iterable fields includes all metric fields and some non-metric fields.

In this note I show that iterability of the field $\mathfrak{A}$, that is,

$$\int \overline{\mathfrak{C}} = \overline{\mathfrak{A}},$$

is also a necessary condition that the class of limited functions over $\mathfrak{A}$ simultaneously satisfy relation (A).

2. THEOREM 1. *If all limited functions defined over $\mathfrak{A}$ satisfy relation (A), then $\mathfrak{A}$ is iterable.*

Consider the contraposite, if $\mathfrak{A}$ is not iterable then not all limited functions over $\mathfrak{A}$ satisfy (A); and we see that to prove Theorem 1 it is only necessary to show there exists a limited function over a non-iterable field which does not satisfy (A). Such a function is as follows:

Example 1. In $\mathfrak{A} = \{x, y\}$, let $0 \leqq x \leqq 1$; for rational x let $0 \leqq y \leqq 1$, and for irrational x let $0 \leqq y \leqq \frac{1}{2}$. Let $f(x, y) = 1$ over this $\mathfrak{A}$.

This $\mathfrak{A}$ is not iterable, for

$$\underline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}} = \tfrac{1}{2}, \qquad \overline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}} = 1,$$

* Presented to the Society (Madison), September 8, 1913.

† These Transactions, vol. 7 (1906), p. 167.

‡ These Transactions, vol. 11 (1910), p. 25.

and therefore $\int_{\mathfrak{B}} \mathfrak{C}$ does not even exist. Further, $f(x, y)$ here does not satisfy relation (A), for

$$\underline{\int}_{\mathfrak{A}} f = 1, \quad \text{and} \quad \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = \tfrac{1}{2}.$$

By combining Theorem 1 above and Theorem 14, p. 35 of my former paper cited above, we get the following important theorem:

THEOREM 2. *In order that all limited functions over $\mathfrak{A}$ simultaneously satisfy relation (A), it is necessary and sufficient that $\mathfrak{A}$ be iterable relative to x.*

3. It is to be noticed that iterability of the field is not necessary in order that a single particular limited function satisfy (A), as the following example shows.

Example 2. Let $\mathfrak{A}$ be defined as in Example 1 above. Let $f(x, y) = 1$ when x is rational and $f(x, y) = 2$ when x is irrational.

Here again $\mathfrak{A}$ is not iterable, but relation (A) is satisfied, for

$$\underline{\int}_{\mathfrak{A}} f = 1, \quad \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f = 1, \quad \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f = 1, \quad \overline{\int}_{\mathfrak{A}} f = \tfrac{3}{2}.$$

4. *Correction.* I wish to call attention to an error in Theorem 8, page 31 of my former paper cited above. This theorem as there stated will not hold for certain fields involving a Pringsheim aggregate* which has a two-dimensional content greater than zero while each linear section parallel to the x-axis (or y) has a linear content equal to zero.

The proof as there given will hold if we restate the theorem as follows:

THEOREM 3. *Let $f(x, y)$ be limited over the limited field $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$ where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are so defined that on each linear section parallel to the x-axis (or y) the points of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are each everywhere dense with respect to the other. Let $f(x, y) \geqq 0$ over $\mathfrak{A}_1$, and $f(x, y) \leqq 0$ over $\mathfrak{A}_2$. Then relation (A) is true.*

Theorem 8 was used in no other place, so this error affects no other part of that paper.

The author is indebted to Prof. R. G. D. Richardson for calling his attention to the above error; also for suggesting the truth and method of attack of Theorem 1 of this paper.

THE UNIVERSITY OF ILLINOIS,
November 7, 1913.

* Pierpont's Lectures on the Theory of Functions of Real Variables, vol. 1, p. 546.

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Page 65. OSWALD VEBLEN. *Decomposition of an n-Space by a Polyhedron.*

Page 66, line 4 of §4, for "consisting" read "whose interior consists."

Page 66, line 6 of §4, for "The" read "A."

Page 66, line 9 of §4, before the word "According" insert the sentence "It is easily seen that a region has at most one boundary."

Page 66, line 4 of §5, between the words "two-dimensional" and "*polygonal*" insert "*polyhedral or a.*"

A polyhedron is defined in §5 as a set of n-dimensional polyhedral regions, together with their boundaries, subject to certain restrictions. These restrictions are not properly stated. Instead of modifying the statements as they stand, I propose to make a change which simplifies the whole matter without any loss of generality.

It is easy to prove, by an argument like that in §7, that the set of points on any polyhedral region together with its boundary can be regarded as composed of a finite number of convex polyhedral regions together with their boundaries. Hence any set of points consisting of a finite number of polyhedral regions and their boundaries may be regarded as consisting of a finite set of convex polyhedral regions and their boundaries. In view of this observation, we replace the definition at the bottom of page 66 and the top of page 67 by the following:

An *n-dimensional polyhedron* is a set of points $[P]$ which satisfies the following conditions: (1) $[P]$ consists of the points in and on the boundaries of a finite number of n-dimensional convex polyhedral regions, $F_1, F_2, \cdots, F_k$, no two of which have a point in common, and which are such that whenever there is an $(n-1)$-dimensional convex region contained in the boundary of each of q F's and containing no point in or on the boundary of any other F, q is an even number; (2) there is no subset of $[P]$ which has the property (1).

This change in the definition renders superfluous the corollary (1) of §16 on page 72, but makes no change in the rest of the paper.

It is perhaps desirable to call attention to the fact that, according to the definition employed in this paper, the boundary of a general region need not be a closed set of points. It does, however, require the boundary of a convex or polyhedral region in a number-plane to be closed.

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